KINEMATICS AND KINETICS OF MACHINERY

A TEXT BOOK FOR COLLEGES AND TECHNICAL SCHOOLS

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PREFACE

This book is the outcome of several years' experience in teaching kinematics and kinetics of machinery at the University of Illinois. For many years this subject was taught from notes prepared by Professor G. A. Goodenough, to which was added an article on the gyroscope by Professor F. B. Seely of the Department of Theoretical and Applied Mechanics. These notes were several times revised by the authors as experience showed where improvements could be made.

In the fall of 1916 the authors undertook, with the consent of Messrs. Goodenough and Seely, to rewrite these notes in textbook form. The present volume is the outcome of that undertaking. The work was interrupted by the war, which took one of the writers into the military service, and imposed on the other such a heavy burden of teaching work that further progress on the book was impossible. In the fall of 1919 the work was resumed and pushed to completion.

The introductory chapter on constraint, the chapter on plane motion, and the chapter on velocities follow closely Professor Goodenough's notes. The chapter on the gyroscope is inserted almost without change as written by Professor Seely. The chapters on accelerations, on inertia forces, on balancing and on governors have been so completely rewritten that little trace remains of the original. The chapters on toothed wheels, on cams, on wrapping connectors, and on critical speeds have been added by the authors.

It is hoped that this volume will fill a need in the curricula of our engineering schools, in that it gives systematic methods of

determining velocities, accelerations, and inertia forces which can be applied to practically all mechanisms. These methods are in the main graphical, the complicated forms of the equations making analytical methods too cumbersome for practical use except in some of the simpler types of machines. If the work is done to a large scale the results should be accurate enough for all practical purposes.

The book is so arranged that it can be readily adapted to short courses as well as to more complete and detailed ones. Thus the chapters on gears, cams, and belts may be omitted where these subjects are taught in the courses in mechanism or design. The chapter on balancing can be profitably studied without the detailed analysis given in the chapters on accelerations and inertia forces. The chapter on critical speeds and parts of the chapters on governors and gyroscopes involve the use of mathematics which is perhaps beyond the range of the average undergraduate. These parts may, however, be of great value to the advanced student who intends to specialize in scientific design. For the benefit of undergraduate students a note on the solution of linear differential equations is appended.

In conclusion the authors wish to extend their thanks to Professor Goodenough for valuable suggestions and criticisms in the preparation of the work.

J. A. D.
A. C. H.

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KINEMATICS AND KINETICS OF MACHINERY

CHAPTER I

MACHINE MOTIONS, PAIRS, LINKS, CHAINS, MECHANISMS

1. Scope of the Subject.—*Mechanics* is that branch of *Physics* which treats of the *motions of material bodies* and the *forces* acting on such bodies. That division of mechanics which deals with motions is called *kinematics*, and that division which deals with forces is called *dynamics*. Dynamics is further divided into *statics*, in which the bodies dealt with are considered to be in equilibrium, and *kinetics*, in which the bodies are acted upon by unbalanced forces.

Mechanics of machinery consists of the study of the application of the laws of mechanics to the parts of machines. The subject may be divided again into *kinematics*, *statics*, and *kinetics*. *Kinematics of machinery* consists of the study of the motions of the parts of machines without regard to the forces accompanying or producing such motions. *Statics of machinery* consists of the study of the forces in machine parts with the assumption that all such parts are in equilibrium, that is, disregarding any forces which may act to produce *accelerations* in these bodies.

In any machine it is impossible that any part shall move indefinitely at a uniform speed and in a straight line. It follows that there must always be forces producing accelerations in the moving parts, or in other words that no moving part can be in equilibrium. The province of *kinetics of machinery* is to take into account the accelerating forces. In many instances the accelerating forces are quite small, and the problems arising in such cases may be treated by static methods. In other cases

the accelerating forces are extremely important. This is true of all high-speed machinery. In the following pages the subjects of kinematics and kinetics of machinery will be treated at length. Static problems will be considered only incidentally.

The study of mechanics of machinery may be approached from two different points of view: (1) the motions and forces in existing machines may be analyzed; and (2) machines may be devised to produce desired motions and forces. It is believed that a thorough study and analysis of existing machines will be of great value to those who later expect to become designers, and it is the purpose of this book to guide students in such study.

In general, a *machine* will be regarded as a system of rigid bodies, so connected that for a given movement of any one part there will be perfectly definite, determinate movement of every other part. The assumption of the rigidity of the parts is equivalent to disregarding any motions due to distortion or vibration of the members. Special cases where such distortions are not negligible, or where flexible links such as belts and ropes are employed, will be given special attention.

2. Constrained Motion. Pairs.—The characteristic of the motion of a machine part, as distinguished from that of a free body, is that every point of the machine element is *constrained* to move in a fixed predetermined path. In order that this may be the case, it is necessary that each machine part must be in contact with one or more other parts. The connections between the parts are called *pairs*. Thus in the ordinary steam engine mechanism, Fig. 1, there are four pairs: (1) between the crank and bearing, (2) between the connecting rod and crank, (3) between the crosshead and connecting rod, and (4) between the crosshead and guides.

3. Properties of Pairs.—Since the purpose of pairs is to constrain the relative motion between the pairing bodies, the first step in analyzing constrained motions is the study of the properties of pairs. The simplest way to constrain a point P to move in a given path would be to cut a slot whose center line is the given path, and place in the slot a block on which is marked the point to be guided. In general, if this block is cut to fit the curvature of the slot in one position, it will not fit in some other position where the curvature of the slot is different, Fig. 2. If, however, the radius of curvature of the slot is constant, the block will fit

equally well in all positions, Fig. 3. If the radius of curvature of the slot is indefinitely increased the circle becomes a straight line, Fig. 4. Rectilinear motion may therefore be regarded as a limiting case of circular motion. It follows that continuous surface contact between pairing bodies is possible only when the relative motion is circular or rectilinear. For paths which have variable

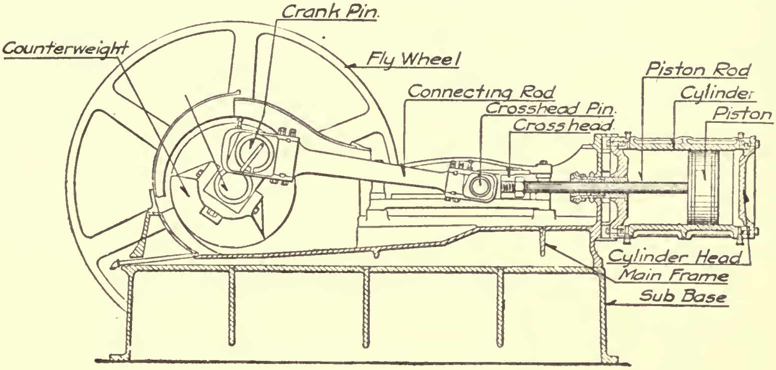


FIG. 1.

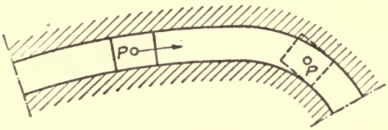


FIG. 2.

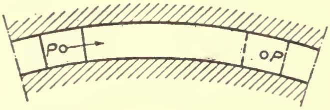


FIG. 3.

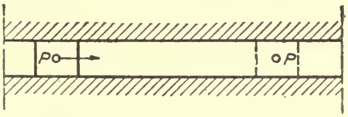


FIG. 4.

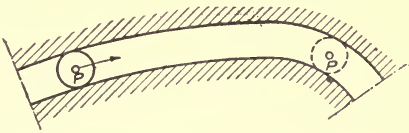


FIG. 5.

curvature only line contact is possible. For example, if the sliding block, Fig. 2, is replaced by a circular pin, Fig. 5, relative motion becomes possible regardless of the curvature of the slot. Pairs which permit surface contact are called *lower* pairs. Those which permit only line contact are called *higher* pairs.

Another characteristic of lower pairing is that not only the point P, but every other point on the block is constrained to move in a similar path. Thus in Fig. 6 if an arm be rigidly attached to

the block, a point P_2 on this arm will travel in a circular path whose center is O, the center of curvature of the slot. On the other hand, in the case of higher pairing the motions of other points on the pin are not constrained. Thus, in Fig. 7, while the point P_1 moves along the slot, the arm attached to the pin may revolve around P_1, and therefore, a point P_2 on this arm may move in any one of an infinite number of paths. That is, the motions of other points attached to the pin are not constrained. A pair which completely constrains the relative motion between the bodies connected is called a *complete* or *closed pair*. One

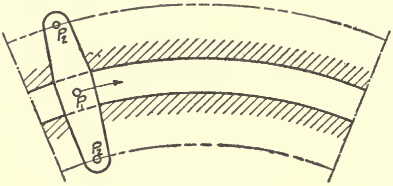


Fig. 6.

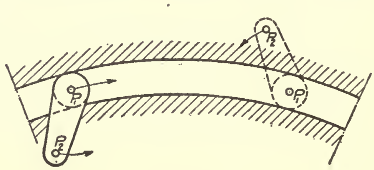


Fig. 7.

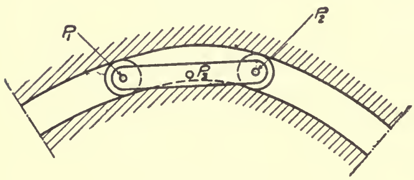


Fig. 8.

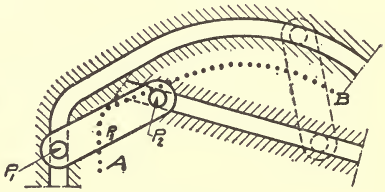


Fig. 9.

which does not so constrain the relative motions of the pairing bodies is called an *incomplete* or *unclosed* pair. All lower pairs are, or may be made into, closed pairs. In general higher pairs are incomplete.

If a rod P_1P_2 has two pins attached so that two points P_1 and P_2 must travel along the center line of the slot, Fig. 8, or along the center lines of the two distinct slots, Fig. 9, then any other point, P_3, of the rod, Fig. 9, is constrained to move in the dotted path AP_3B. Thus in general two unclosed pairs furnish the same degree of constraint as one closed pair. (A more rigorous treatment of this subject will be found in Art. 28.)

Referring again to Fig. 6, it is evident that the circle about O might be completed, and the block expanded to fill the whole

of this circle, without in any way altering the character of the motion, as shown in Fig. 10. Thus an ordinary pin-and-eye connection, as for example, between the crank and connecting rod of a steam engine, constitutes a lower pair.

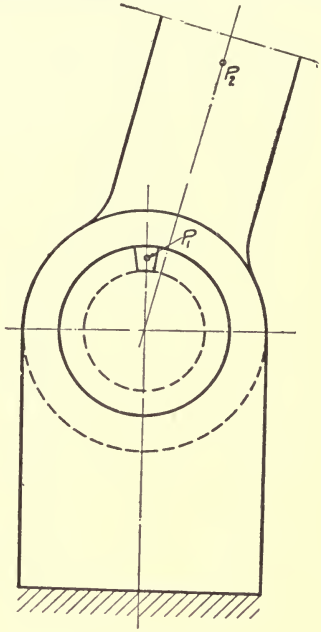


Fig. 10.

A third characteristic of lower pairs may be described as follows: if one of the pairing bodies is held stationary and the other caused to move, a point on the moving body will trace a definite path; if the arrangement is now reversed, the second part being held stationary and the first being moved, a point on the moving body will trace a second path similar to the first. With higher pairing such paths will in general be dissimilar.

Finally it may be pointed out that lower pairs are easily constructed by means of standard machine tools such as lathes, planers, etc., while higher pairs often require special machines such as gear cutters. The properties of lower and higher pairs may be summarized as follows:

	Examples.	Contact.	Constraint.	Relative Motion.	Construction.
Lower pairs	Pin and eye Slide	Surface*	Complete	Same path traced by moving point whichever body is held stationary	Lathe, planer, etc.
Higher pairs	Gears Cams	Line	Incomplete	Different paths traced by moving point according to which body is held stationary	Gear, cutters, etc.

* In some special cases lower pairs may have line contact only. An example is a round shaft rotating in a square bearing. In such cases extra material may be added to give surface contact without in any way altering the character of the motion. As the relative motion is the important consideration such pairs are classified as lower.

4. Pairing Elements.—The geometrical forms placed upon two bodies so that they may be connected by a pair are called *pairing elements.* Thus the cylindrical surface of the wrist pin of an engine, and the inside surface of the brasses of the connecting rod are pairing elements. The surfaces of the crosshead and guides, the faces of two gear teeth in mesh, or of two cams in contact, furnish other examples.

5. Inversion of Pairs.—In the case of lower pairs the solid and hollow elements may be interchanged without changing the character of the relative motion. For example, instead of having a pin in the crosshead of an engine fitting into an eye in the connecting rod, a pin might be attached to the rod fitting into a hole in the crosshead. Instead of having a moving piston in a stationary cylinder a moving cylinder might slide over a fixed piston,

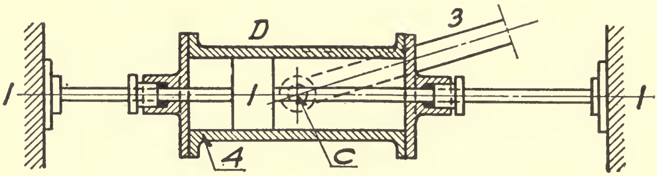


Fig. 11.

the connecting rod being attached to the moving cylinder as shown in Fig. 11. The process of exchanging the hollow and solid elements of a pair is called *inversion* of the pair.

The size of the pairing elements has no effect on the relative motion of the bodies connected by the pair. Thus, if a rod, Fig. 12, has a hole which fits over a stationary pin, the path of any point on the rod is a circle whose center is the center of the pin, regardless of the diameter of the pin. Sometimes by simply expanding pairing elements the appearance of a machine may be changed beyond recognition without altering in the slightest degree the character of the motions. For example, a simple machine is shown in Fig. 13 consisting of four bars connected by turning pairs. By simply expanding the pairing elements this machine may be made to assume the different forms shown in Figs. 14 to 19[1] *without changing* in any way the relative motions. The machine

[1] See Kennedy's Mechanics of Machinery, pp. 293 to 402.

shown in Fig. 14 is precisely the same as that in Fig. 13, except that the pair A has been expanded till the bar AB has been changed to a disk whose circumference is one of the pairing elements.

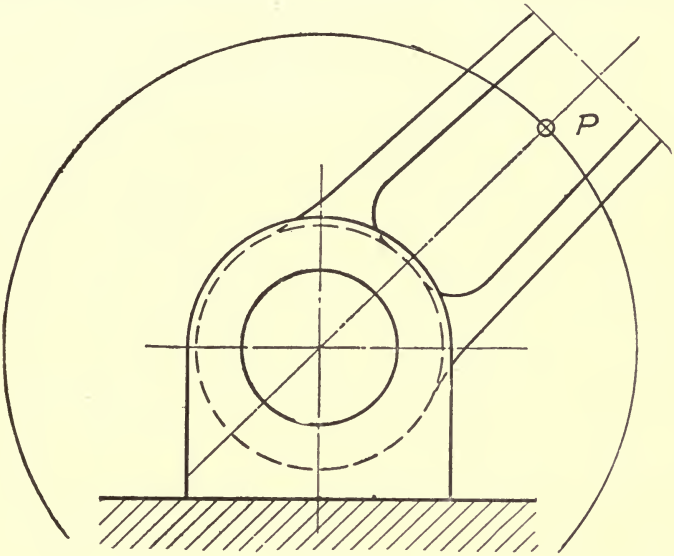


FIG. 12.

In Fig. 15 the same bar AB has been changed to a circular disk, this time by the expansion of the pair B. In this case the bar AB has been changed to an eccentric; hence comparing Figs. 13 and 15 it is evident that *an eccentric is equivalent to a crank whose length is equal to the eccentricity.* In Fig. 16 the pair D is enlarged. Evidently the motion of DC will be equally constrained if the sides of the circular disk are cut away, leaving the zone shown by dotted lines. Another modification is shown in Fig. 17, where the link CD is given the form of an annular ring instead of a disk. The next step is to cut away all of the ring except a small block containing the pair C. Finally since by the rotation of the crank AB the pair C

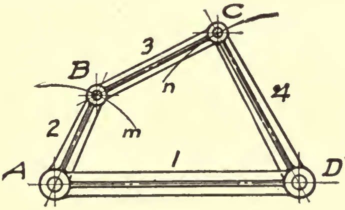


FIG. 13.

is made to reciprocate in a limited circular arc, we may as well shorten the slot in which the block moves to the length actually required, as shown in Fig. 18. *Comparing Fig. 18 with Fig. 13 we see that the lower pair shown in Fig 18 is equivalent to a turning pair whose center is the center of curvature of the slot.* The block containing the point *C* is equivalent to a bar of length equal to the radius of curvature of the slot and pivoted at the center of curvature. By increasing the radius of the slot the equivalent of very long links may be obtained without increasing the dimensions of the machine. A sliding pair is thus equivalent to a turning pair whose center is at an infinite distance. By a series of modifications similar to those described, the mechanism shown in Fig. 19

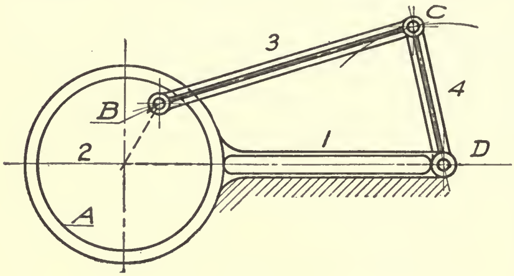


FIG. 14.

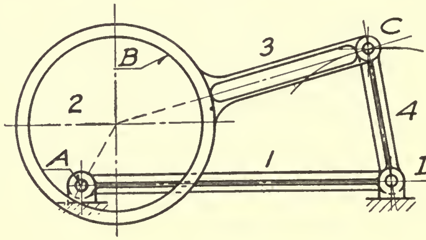


FIG. 15.

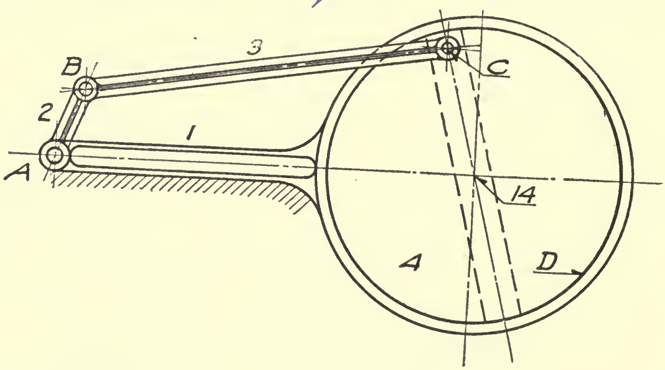


FIG. 16.

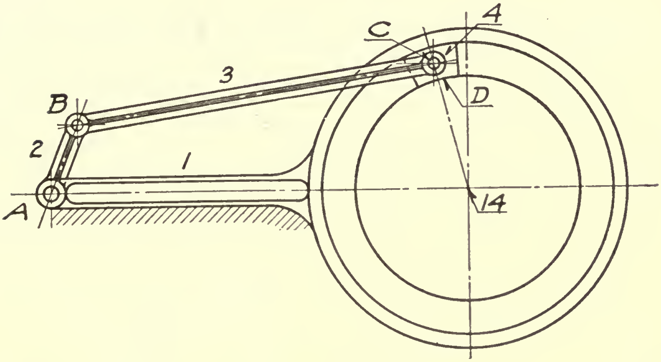


FIG. 17.

is obtained. All of the arrangements shown in Figs. 13 to 19 are kinematically identical.

6. Lower Pairs with Multiple Contact.—It frequently occurs in machine construction that a single lower pair may have several contact surfaces. Thus the shaft of an engine runs in two bearings, and both the crosshead and piston are provided with sliding pairs. A single sliding pair may have any number of contact surfaces provided these surfaces are parallel, and a single turning pair may have any number of journals and bearings provided that they all have the same axis. In the analysis of mechanisms, it is important that the student guard against the common error of counting each contact surface a pair.

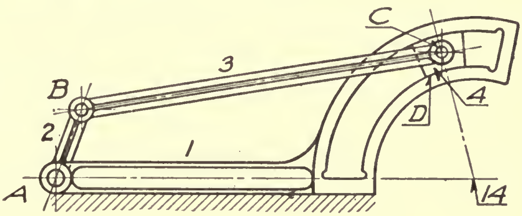


FIG. 18.

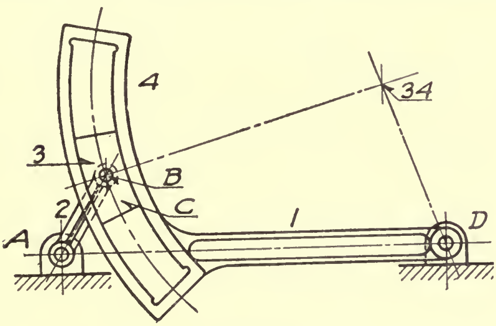


FIG. 19.

7. Links.—A body provided with two or more pairing elements, so that it may be connected to at least two other bodies, is called a *kinematic link*, or simply a link. A body with two pairing elements is called a *binary* link, one with three pairing elements a *ternary* link, one with four pairing elements a *quaternary* link.

In machines having more than two members each link must have at least two pairing elements in order that the motions of all links may be constrained. It should be noted that the term link applies to all parts of a machine which are rigidly fastened together so that there can be no relative motion. For example, in the steam engine, Fig. 1, we may identify the following links:

1. Cylinder and cylinder heads, base plate, foundation, main bearing, crosshead guides, etc.
2. Piston and rings, piston rod, crosshead, wedges, bolts, etc.
3. Connecting rod, brasses, straps, wedges, etc.
4. Crank shaft, crank arm, crank pin, counterweight, flywheel, etc.

The motions in the machine depend entirely on the relative positions of the pairs, and not at all on the size and shape of the links. In particular the stationary link may be very large and heavy, and extra material may be added indefinitely, without in any way affecting the motions, so long as the positions of the pairing elements are unchanged.

8. Chains.—If a number of links are provided with suitable pairing elements they may be connected by joining the elements. If this is done so that each element is provided with a mate and none left unpaired, the resulting structure is called a *kinematic chain*, or simply a chain. Such chains are divided into three classes as follows:

1. Constrained chains, in which relative motions of the links are possible, and where such motions are completely predetermined by the character of the pairs.
2. Locked chains, in which no relative motion is possible.
3. Unconstrained chains, in which the relative motions are indeterminate.

In Figs. 20 to 22 are shown chains consisting of three, four, and five binary links respectively. In Fig. 20 evidently no relative

motion is possible and the chain is therefore locked. Thus if link 1 is held fast, say to the paper, the axis of the pair *B* considered as part of link 2 must move, if at all, in an arc about *A* as a center. Likewise considering the axis of *B* as a point on link 3 it must move, if at all, in an arc about *C* as a center. But, since the point cannot move simultaneously in two different paths, it cannot move at all, and the chain is therefore locked.

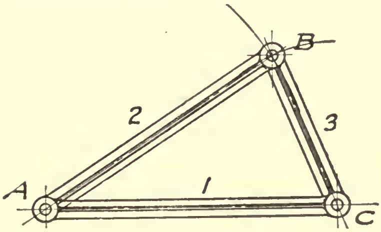


Fig. 20.

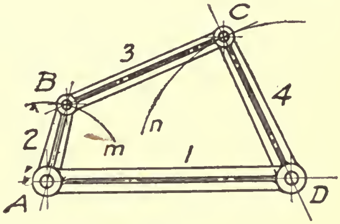


Fig. 21.

In Fig. 21 if link 1 is held fast and link 2 is moved, the only possible motion of link 2 is a rotation about *A* as a center, and point *B* must therefore move in a circular arc. Also point *C* must move in a circular arc about *D* as a center. But, since *B* and *C* must remain at a constant distance apart, for each position of *B* there is in general only one position which *C* can take. The motions of all points on any of the links are therefore completely constrained.

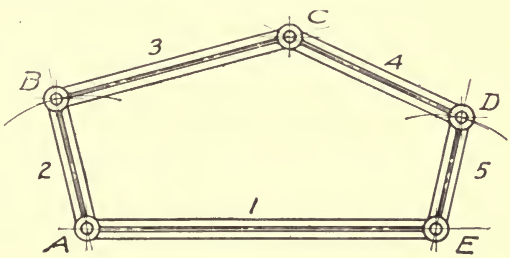


Fig. 22.

The five-link chain, Fig. 22, is evidently unconstrained. As in Fig. 21, if link 1 is held stationary two points *B* and *D* must move in circular arcs about *A* and *E* as centers. But, since points *B* and *D* need no longer remain a constant distance apart, they may be moved independently of each other, and the point *C* may be made to travel in an infinite variety of paths. The motions are therefore indeterminate, and the chain is unconstrained.

Each of the three types of chains is useful for particular purposes. Locked chains are not used in machines, since the parts of a machine are supposed to have relative motions. They are, however, extensively employed in bridges, roofs and other structures.

In the great majority of machines constrained chains are employed. Unconstrained chains are used only in cases where a greater degree of freedom is desired than is afforded by constrained chains. The following are examples:

1. In a steam engine with shaft governor the motions of each point on the piston, crosshead and connecting rod can be accurately determined from those of the crank; but the governor mechanism and the slide valve may be given motions independent of that of the crank. The engine mechanism as a whole is therefore unconstrained.
2. Reversing gears for steam engines may be given motions independent of the motions of the rest of the engine, and may thereby be made to influence the motions of the valves.
3. In the Corliss engine the movement of the valve, when the hook is released, is entirely independent of the rest of the mechanism.
4. In many machines springs or other flexible members are introduced to allow extra movements when the forces in the machine become excessive.

It will be noted that in practically all these cases the machines run as if constrained. In the examples cited, in case:

1. The engine runs as if constrained except when the governor is actually shifting:
2. Except when the engineer is moving the reverse lever;
3. Part of the time the motion of the lever is constrained, the atmospheric pressure supplying the required force to move the valve after the hook releases.
4. Except when unusually heavy forces occur;

In most cases the unconstrained chain is entirely useless. Practically all the discussions in the following pages will apply to constrained chains.

9. Mechanisms.—The step from the constrained chain to the mechanism is a simple one. Suppose one of the links of a constrained chain to be held fixed relative to the ground or some other standard. Then since each of the links has a constrained motion relative to the fixed link, it also has a constrained motion relative to this standard. The chain with one of its links fixed is called a mechanism. Since any one of the links may be chosen as the fixed link, *a chain has as many mechanisms as it has links.*

10. Skeleton Links.—As the motions of a constrained chain depend only on the location of the pairing elements and not on the size and shape of the links, we may represent the links by simple geometrical figures. Thus a binary link may be represented by a straight line having pairing elements at its ends, a ternary link by a triangle, a quaternary link by a quadrilateral, etc., having the pairing elements at the vertices. Such *skeleton* links are represented in Fig. 23. In this figure an element of a sliding pair is represented by a small block, an element of a turning pair by a small circle, and an element of a higher pair either by a small triangle or by the profile of the pairing element. In this way mechanisms may be represented by simple figures, all unnecessary lines being eliminated.

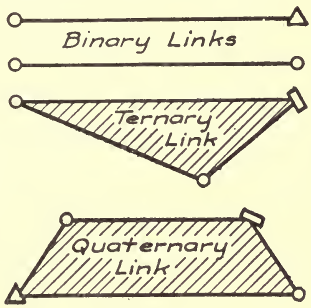


Fig. 23.

11. Formation of Constrained Chains.—As machines are usually constrained chains, two problems now arise: (1) To study the methods of connecting a series of links so as to form constrained chains; (2) to determine the conditions which must be satisfied in order that a chain shall be constrained.[1]

To build up a constrained chain from binary and ternary links, Fig. 24, start with the links 1, 2, 3, and 4, giving a four-link chain with pairs A, B, C, and D. Add an element E to link 3 and an element F to link 4, thus making links 3 and 4 ternary; join E and F by means of two binary links 5 and 6 having a common pair

[1] In this and the following discussions only mechanisms in which all points move in parallel planes are considered.

G. The result is a six-link chain with two ternary and four binary links. The chain is constrained, for points *E* and *F* have different motions in known paths, and this fact imposes upon *G* a determinate motion. If *E* and *F* are connected by one link, the chain is locked, and if more than two links are used the chain is unconstrained. This chain is known as the Watt chain, because with link 3 fixed it gives the mechanism of the Watt Beam Engine.

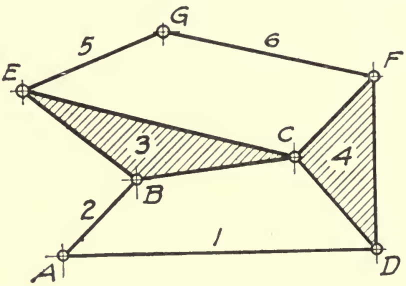


Fig. 24.

If the element *E* is added to link 2, making the non-adjacent links 2 and 4 ternary links, the result is a second six-link constrained chain as shown in Fig. 25. This arrangement is known as the Stephenson chain.

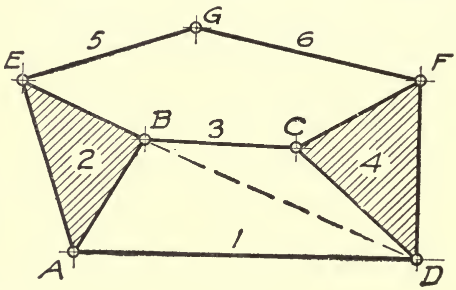


Fig. 25.

In a ternary link two of the elements may coincide. Suppose that the elements *B* and *E* of link 3, Fig. 24, thus fall together, the resulting chain is shown in Fig. 26. This chain is also obtained by making *B* and *E* of link 2, Fig. 25, coincident. Furthermore, elements *F* and *D* of link 4 may be made coincident, thus obtaining the chain shown in Fig. 27, in

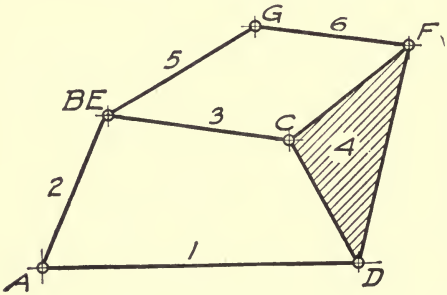


Fig. 26.

which all the links are binary. It is to be noted that neither F nor G may be made to coincide with C, for in that case three of the links would form a locked triangle and the chain would in reality have only four links.

A joint such as B, E, or D, F, Fig. 27, which is made up of three elements, is called a *ternary* joint, one made up of four elements is called a *quaternary* joint, and so on. The ordinary joint of two elements is a *binary* joint. Evidently a *ternary* joint is equivalent to two binary joints; a quaternary joint to three binary joints; and a joint of i elements to i-1 binary joints.

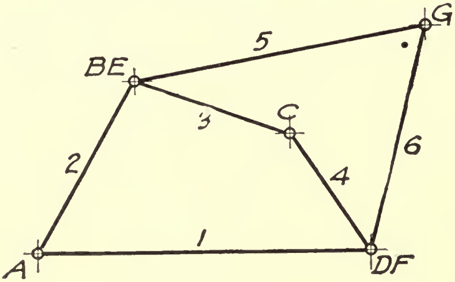


FIG. 27.

Starting again with the 4-link chain, let links 2, 3, and 4 be made ternary by the addition of elements E, F, and G, Fig. 28.

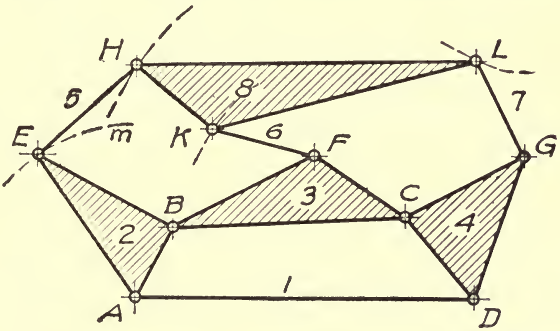


FIG. 28.

At these points connect three binary links 5, 6, and 7, and pair these with a fourth ternary link 8. The result is an 8-link chain with 4 binary and 4 ternary links. That this chain is constrained is established by the following reasoning. Consider link 1 fixed and let the elements of the pair E be separated. Denote the element of pair E that belongs to link 2 by E_2, and the element that belongs to link 5 by E_5. If link 2 be moved, E_2 must move in

a curve m and the joints F and G will likewise move in definite paths. Suppose E_2, F and G to be moved to some position; then keeping F and G at rest, links 6, 7, and 8 may be given a constrained motion and consequently, H, K, and L will move in definite paths as shown. At the same time E_5 may be made to follow the path m and therefore may be brought into coincidence with E_2. These two operations may be repeated indefinitely; E_2, F, and G are first moved; then held fixed while K, L, and H are moved and E_5 is made to coincide with E_2. But these successive motions may be made as small as we please and may

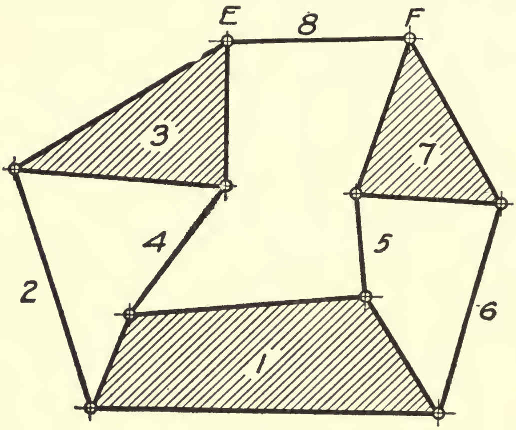


FIG. 29.

take place at the same time instead of in succession. In this case E_2 and E_5 will remain together and may be joined to form a pair. Now since for every position of E, F, and G, there are definite positions for H, K, and L in which E_5 can be made coincident with E_2, it follows that if E_5 and E_2 remain together H, K, and L will move in definite paths. This fact proves the constrainment of the chain.

Other 8-link constrained chains are shown in Figs. 29 and 30. In the chain Fig. 29 two 4-link chains with a common link are joined by the link 8. If link 1 is held fixed the points E and F are evidently constrained to move in definite paths; hence if E and F are joined by the link 8, that link must have a definite motion relative to link 1.

Figs. 31 and 32 show chains with 10 and 12 links, respectively. The constrainment of the 12-link chain is easily proved, there being no less than three 4-link cells. It is not easy, however, to determine whether the chains shown in Figs. 30 and 31 are constrained, as all the cells are bounded by five or more links. In general, it is difficult or impossible to determine by inspection whether a chain of more than six links is constrained. There is, however, a simple relation between the number of links and the number of joints in a constrained chain that may be used as a *criterion of constraint.* This relation is reduced in the next paragraph.

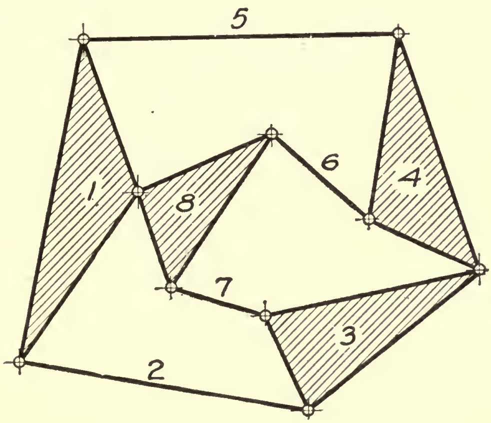


FIG. 30.

12. Criterion of Constraint.—In order that several points *A*, *B*, *C*, *D*, etc., Fig. 33, may occupy always the same relative position a certain number of conditions are necessary. With two points *A* and *B* there is one condition required, namely that the points shall always be the same distance apart or that the length of the line joining *A* and *B* shall be constant. If a third point *C* is added to the system, two more conditions are required to

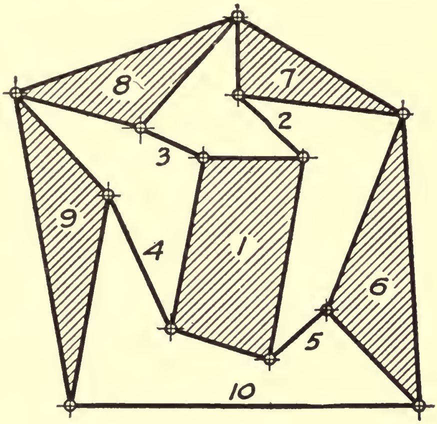


FIG. 31.

fix C relatively to A and B. These may be the lengths of the lines AC and BC or the angles α' and α'', which these lines make

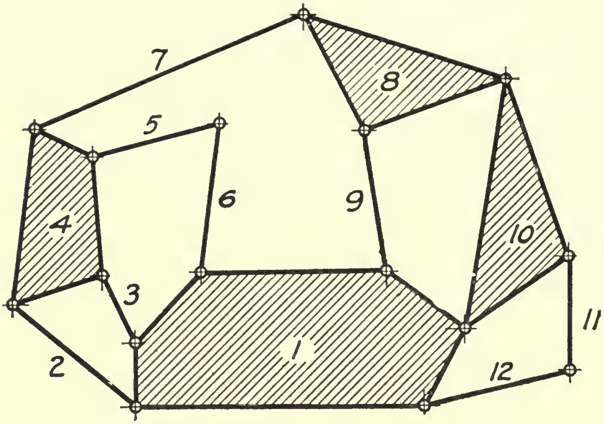


FIG. 32.

with AB. Three points are therefore fixed by three conditions. Similar reasoning shows that a fourth point D requires two more conditions, and each additional point two additional conditions. In general, to fix the relative positions of n points $2n-3$ independent conditions are necessary.

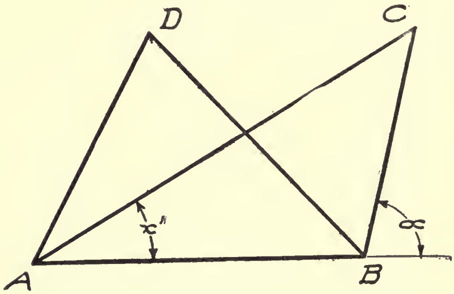


FIG. 33.

In a chain the joints are the characteristic points whose relative positions are to be investigated; and the necessary conditions are furnished by the links. Consider the chain shown in Fig. 25. If the joints B and D are connected by a rigid link, as shown by the dash line, the chain is locked. Thus adding one condition to a constrained chain locks it, and conversely, the removal of one condition in the case of a locked chain, permits constrained motion, that is, gives the chain one degree of freedom.

Let J = number of joints in the chain;
C = number of conditions furnished by the links.

We have seen that $2J-3$ conditions just lock the chain; hence $2J-4$ conditions leave the chain constrained. Expressed in algebraic language:

If $C=2J-3$, the chain is *locked;*
$C=2J-4$, the chain is *constrained;*
$C \overline{\overline{<}} 2J-5$ the chain is *unconstrained.*

Each link of the chain furnishes a number of conditions depending on the number of its pairing elements.[1] A binary link gives one condition, the distance between the joints connected by it; a ternary link gives three conditions, the three required to fix the relative positions of its three elements; a quaternary link contributes the five independent conditions that fix its four elements. In general, a link with k pairing elements contributes $2k-3$ independent conditions.

Let N = total number of links in the chain;
J = total number of joints in the chain;
E = total number of pairing elements;
n_2, n_3, n_4, etc. = respectively, the number of links of 2, 3, 4, etc., elements;
m_2, m_3, m_4, etc. = respectively, number of binary, ternary, etc., joints;
C = number of conditions furnished by the links.

Then

$$E=2n_2+3n_3+4n_4. \qquad (1)$$

Also

$$E=2m_2+3m_3+4m_4. \qquad (2)$$

$$N=n_2+n_3+n_4 \qquad (3)$$

$$J=m_2+m_3+m_4. \qquad (4)$$

$$C=n_2+3n_3+5n_4+7n_5. \qquad (5)$$

[1] In this paragraph it is assumed that the chain contains only lower pairs. The case of higher pairing is considered in the next article.

If this chain is constrained,

$$C=2J-4.$$

Hence

$$n_2+3n_3+5n_4+7n_5 \ldots =2J-4. \quad \ldots \quad \text{(I)}$$

This equation is the criterion of the constrainment of the chain.

Adding equations (3) and (I),

$$2n_2+4n_3+6n_4+8n_5 \ldots =2J-4+N. \quad \ldots \quad (6)$$

Subtracting (3) from (1),

$$n_2+2n_3+3n_4+4n_5 \ldots =E-N. \quad \ldots \quad (7)$$

Comparing (6) with (7).

$$2(E-N)=2J-4+N$$

Or

$$E-J=\tfrac{3}{2}N-2. \quad \ldots \quad (8)$$

If the chain has only binary joints,

$$E=2n_2=2J$$

and (8) reduces to

$$J=\tfrac{3}{2}N-2. \quad \ldots \quad \text{(II)}$$

Equation (II) is the form in which the criterion is generally used. Even when a chain has ternary and quaternary joints, we may consider each ternary joint replaced by two binary joints, each quaternary joint by three binary joints, and so on; and use the form (II). From the form of (II) it appears that a chain with only lower pairs cannot in general have an odd number of links.

As an exercise, let the student apply both criterions to each of the chains shown in Fig. 20 to 32.

13. Criterion of Constraint with Unclosed Pairs.—The term *joint* is restricted to the connection of two links by a complete or closed pair. The mere point of contact which ordinarily occurs in higher pairing is not a joint in this sense. In chains with higher pairs, therefore, there may be links with only one joint. Gear wheels and cams are nearly always links of this character. In conformity with the notation of the preceding paragraph, the

number of such links is denoted by n_1. It has been shown that a link with k joints gives $2k-3$ conditions; following this rule each link with one joint gives $2\times1-3=-1$ conditions. The interpretation of this statement is simple: such a link with its one joint furnishes no conditions at all, but instead requires one condition, namely that it shall be in contact with or have higher pairing with some other link.

In addition to the conditions furnished by the links carrying two or more joints, each unclosed pair furnished one condition, namely that some pair of links have contact. If H denotes the number of unclosed pairs, then these higher pairs furnish H conditions. The total number of conditions furnished is therefore:

$$C=-n_1+n_2+3n_3+5n_4+\ \ldots\ +H \quad . \quad . \quad . \qquad (1)$$

As before, the number of conditions necessary to render the chain constrained is $C=2J-4$; hence

$$-n_1+n_2+3n_3+5n_4\ \ldots\ +H=2J-4. \quad . \quad . \qquad (\mathrm{III})$$

Equation (III) is the most general form of the criterion for chains with unclosed pairs. A form analogous to Equation (II) may be derived as follows:

$$E=n_1+2n_2+3n_3+4n_4. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$N=n_1+n_2+n_3+n_4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

$$2J-4-H=-n_1+n_2+3n_3+5n_4+7n_5. \quad . \quad . \qquad (4)$$

Combining Equations (2) and (3), and Equations (3) and (4):

$$E-N=n_2+2n_3+3n_4+4n_5\ \ldots$$

$$2J-4-H+N=2n_2+4n_3+6n_4+8n_5\ \ldots$$

Therefore

$$2(E-N)=2J-4-H+N, \quad . \quad . \quad . \quad . \qquad (5)$$

or

$$E-J=\tfrac{1}{2}(3N-4-H). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

If all the joints are binary, $E=2J$; and therefore

$$J=\tfrac{1}{2}(3N-4-H) \quad \text{or} \quad J+\tfrac{1}{2}H=\tfrac{3}{2}N-2^{1}. \quad . \quad . \qquad (\mathrm{IV})$$

[1] The criterion here developed is for plane mechanisms only; that is, mechanisms in which all points move in parallel planes.

In the rarer cases in which higher pairs are closed pairs, they should be counted among the joints J rather than among the unclosed pairs H.

14. Application of Criterion.—If the criterion in this form be applied to any mechanism the left-hand member may be equal to, larger than, or smaller than the right-hand member. If the two are equal, the mechanism is constrained. If the left-hand member is the larger, the mechanism is locked; if the smaller, the mechanism is unconstrained. The left-hand member may be taken as representing the number of restrictions imposed on the motions of the links, and the right-hand member as the number of restrictions necessary to constrain their motion. If the restrictions are too numerous no motion is possible, if too few there is no constraint. It will be noted from the form of the criterion that an unclosed pair imposes on the motion of the pairing links only half the constraining effect of a closed pair. (Compare Art. 6.)

15. Exceptional Cases.—In some cases the test for constraint indicates that a mechanism is locked, while we find that it is actually free to move. In every such case a study of the motions reveals that some point is constrained to move in the same path by two separate devices, either of which would alone be sufficient. The *Davis engine*, Fig. 34, is an example. Two cylinders are placed at right angles to each other as shown. The long piston rods 2 and 3 are guided at both ends and are joined by turning pairs to the connecting rod 4. The crank 5 is joined to link 4 at A, the middle point of the connecting rod. We have here five links and six joints, namely 12, 13, 15, 24, 34 and 45.

The criterion of constraint gives $6 \doteq \frac{3}{2} \times 5 - 2 = 5\frac{1}{2}$ and the machine appears to be locked.

A study of the mechanism shows that if the crank 5 were omitted the point A would still be obliged to travel in a circle about O as a center, and that therefore the crank really adds nothing to the constraint. If the crank were attached to the connecting rod at any other than the middle point, the mechanism would be locked as every point of the rod such as B is constrained to move in an ellipse. Since a crank attached at B would constrain point B to move in a circle the mechanism would be locked.

In a few cases machines which are indicated as locked, are enabled to run by allowing a small amount of lost motion or by

slightly bending some member. An example is shown in Fig. 35. In this mechanism A as a point on link 5 must move in a horizontal

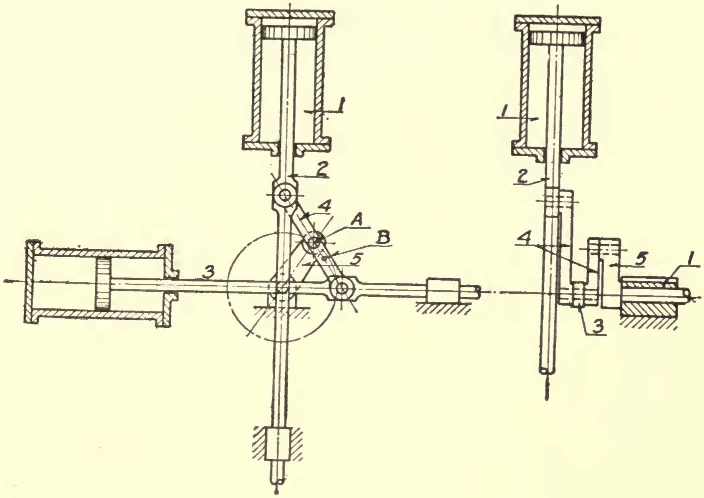


FIG. 34.

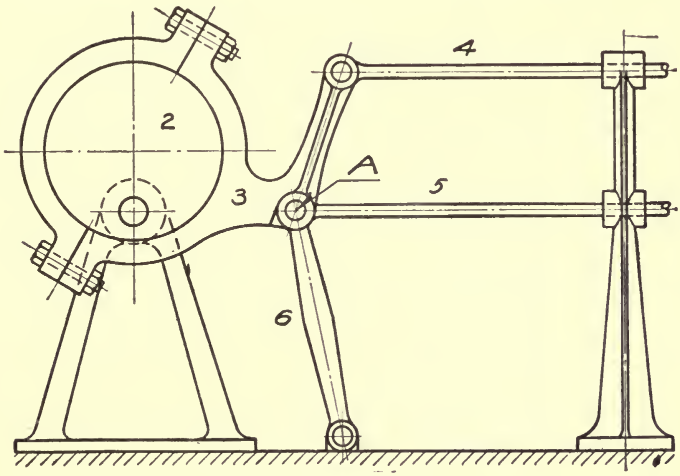


FIG. 35.

line. As a point on 6 it must move in an arc of a circle. The machine is therefore apparently locked. However, a small amount

of lost motion in the pair between links 1 and 5, or a slight bending of link 5 will allow the machine to move.

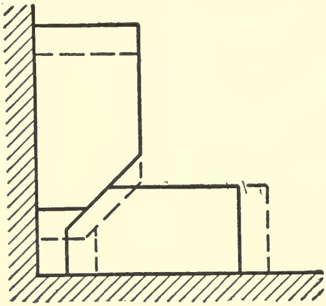


Fig. 36.

In one very simple mechanism the criterion as stated is at fault. In the wedge shown in Fig. 36, we have three links and three sliding pairs. The criterion gives:

$$3+0=\tfrac{3}{2}\times 3-2=2\tfrac{1}{2},$$

and the machine is apparently locked, although it is evident that movement is possible. A chain containing only sliding pairs possesses an extra degree of freedom of motion as compared with other chains having the same number of links and joints.[1]

As examples in the use of the criterion let the student test for constraint the skeleton mechanisms shown in these pages, and any other plane mechanisms which present themselves. In every case where the criterion is not satisfied, study the motions carefully and try to discover the cause.

16. Analysis of Mechanisms.—It has been seen that the motions of the links depend entirely upon the character and location of the pairing elements, and that consequently a binary link may be conveniently represented by a straight line connecting the elements, a ternary link by a triangle, etc. The analysis of a mechanism consists of identifying the links, and noting with what other links each one pairs. Then each link may be represented by its skeleton form—straight line, triangle, etc.—and the pairing elements properly marked. As an example take the Stephenson link motion Fig. 37. Here we can identify eight links:

(1) Frame of engine.
(2) Shaft and two eccentrics.
(3) Eccentric rod.

[1] The complete criterion of constraint is given by

$$J+\tfrac{1}{2}H-\tfrac{1}{2}S=\tfrac{3}{2}N-2,$$

where S is the number of complete cells containing only sliding pairs. See Kinematics of Machinery, by A. W. Klein.

(4) Eccentric rod.
(5) Slotted link.
(6) Hanger.
(7) Block moving in slotted link.
(8) Slide valve and stem.

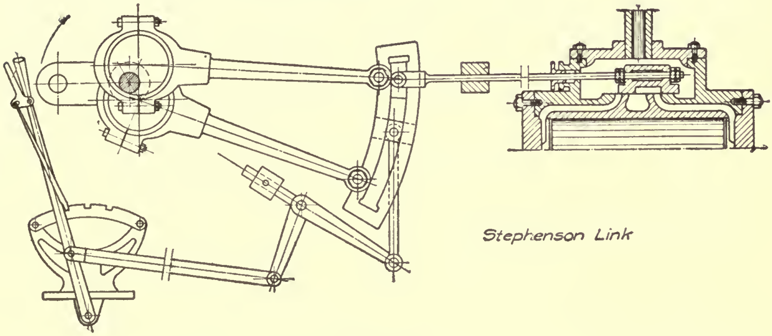


Fig. 37.

These links are shown in skeleton form in Fig. 38. Taking the links in order we see that link 1 is a ternary link pairing with links 2, 6 and 8. It is therefore represented by a triangle. The joints with links 2 and 6 are turning while that with link 8 is sliding. Link 2 has three turning pairs with links 1, 3, and 4. Link 3 has turning joints with links 2 and 5, and the same is true of link 4. Link 5 has four turning joints with links 3, 4, 6, and 7. Link 6 has turning joints with links 5 and 1. Link 7 has turning joints with links 5 and 8. Link 8 has a turning joint with link 7 and a sliding joint with link 1.

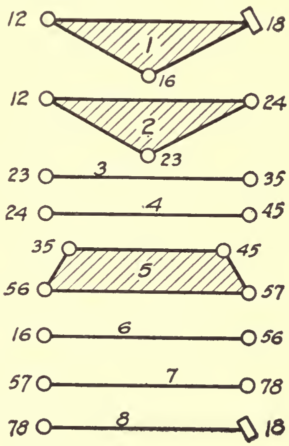


Fig. 38.

Now, if desired, the links may be connected to form a chain, which then represents the machine in its simplest or skeleton form, as shown in Fig. 39. This chain may readily be shown to be constrained by applying the criterion.

Often when analyzed in this way, machines which externally are very different in appearance are found to be built on the same skeleton chains.[1] For example, if we omit links 7 and 8, the chain just described is identical with that of the Wanzer needle bar, Fig. 106. If link 2 or link 5 is held stationary, it becomes the chain for the Atkinson gas engine, Fig. 165; the Blake stone crusher, Fig. 147; or the shaper, Fig. 54.

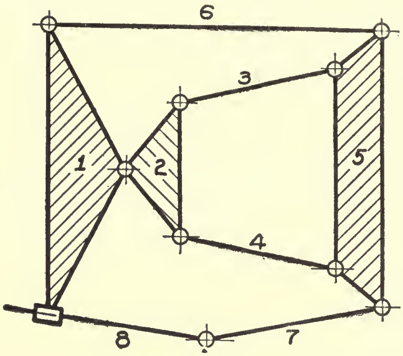


FIG. 39.

17. Inversions of Mechanisms.—It has been seen that by holding different links of a chain fixed in turn, various mechanisms are formed. This process is called *inversion of mechanisms.* For example, in Fig. 40 if link 1 is held stationary and gear 2 made to revolve, the compound gear 3 will revolve and drive gear 4 as in the ordinary back gear of a lathe. If, however, one of the gears, say gear 2, is held stationary and link 1 allowed to revolve, the result is an epicyclic or planetary gear train, which is an inversion of the ordinary gear train. Similarly the rotary engine, the oscillating engine, the

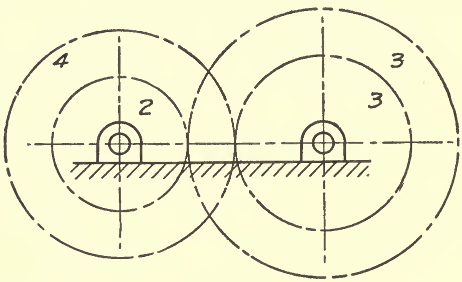


FIG. 40

[1] The number of constrained chains which can be formed from links having only lower pairs is given by the following table:

Number of links.	Number of Chains.
6	2
8	14
10	228
12	4000

See Klein's Kinematics of Machinery.

Whitworth quick return and the crank-shaper quick-return motions are all inversions of the ordinary slider crank or steam engine mechanism.

EXERCISES

1. The slider crank chain shown in Fig. 1 is the mechanism of the ordinary steam engine when the link 1 is fixed. Transform this mechanism into three

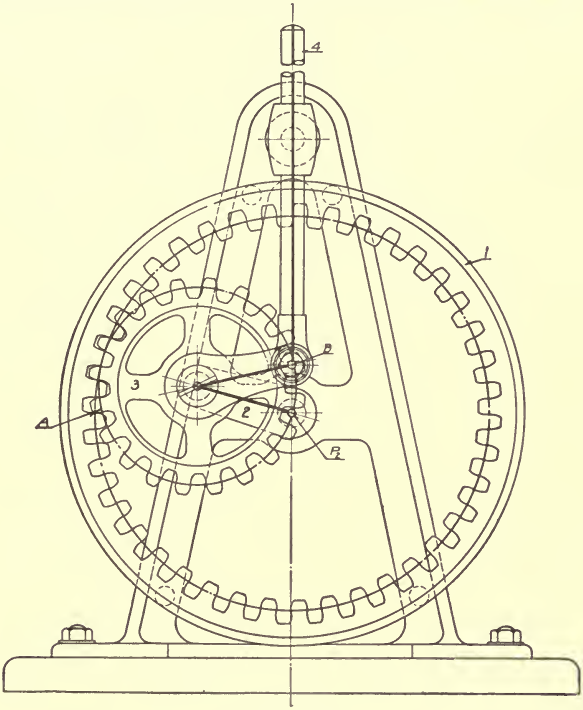


Fig. 41.

other steam engines by successively fixing links 3, 2, and 4. Make the length of link $2 = 1$ inch and that of link $3 = 3\frac{1}{2}$ inches in each case. Be sure that in each case:

Link 1 joins sliding pair D to turning pair P;
Link 2 joins turning pair P to turning pair B;
Link 3 joins turning pair B to turning pair C;
Link 4 joins turning pair C to sliding pair D.

In the solution of this problem it will be desirable in one case at least to interchange the elements of one or more pairs. The exchange of the solid and hollow elements of a lower pair does not affect the relative motion of the links

connected by the pair, and is always permissible. Take, for example, the pair B, Fig. 1; usually the crank 2 carries a pin which fits into an eye in the connecting rod 3; sometimes, however, the rod 3 carries the pin and the crank 2 the eye, this construction being common in the cheaper classes of machinery. In the case of the sliding pair D, link 1 usually carries the hollow enclosing element, i.e., the cylinder and guides; this pair may likewise be inverted by making 4 carry the enclosing element, as shown in Fig. 11. The inversion is inexpedient when 1 is the fixed link, but is useful when 4 is taken as the fixed link, as it makes the lighter element the moving part.

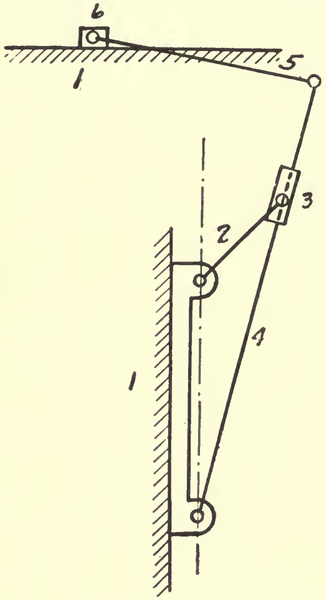


Fig. 42.

Conditions to be observed:

(1) Provide each engine with inlet and outlet for steam, but do not design valve gear.

(2) Provide each engine with means of communicating motion to a shop shaft, but do not add extra links (an extra belt may be used when 4 is the fixed link).

(3) See that no link interferes with the proper motion of any other link.

(4) See that the parts are given such dimensions that the required motion is possible.

(5) Avoid arrangements that give heavy masses reciprocating motions, and balance rotating parts as far as possible. First make free-hand sketches in note books and submit before beginning finished drawing. Make the drawing neat and workmanlike, but do not waste time drawing small details such as keys, bolts, etc.

2. Invert the hypocyclic straight line motion Fig. 41 so that the smaller gear 2 revolves eccentrically around a fixed center, and the larger gear reciprocates in a straight line without turning.

3. Invert the shaper quick-return mechanism Fig. 42 so as to obtain a quick-return mechanism in which there are two cranks, one of which may rotate at constant speed and the other at a variable speed.

Note.—This inversion may be accomplished by simply changing dimensions, retaining link 1 as the stationary link.

Work independently in solving these problems.

REFERENCES

Klein, Kinematics of Machinery; Weisbach, Machinery of Transmission, Introduction; Rankine, Machinery and Mill Work; Kennedy, Mechanics of Machinery.

CHAPTER II

MOTION OF RIGID BODIES

18. Plane Motion of Rigid Bodies.—Plane motion of a body is defined as motion such that all points of the body move in parallel planes. For example, a shaft rotating in a bearing or a slide moving between parallel guides has plane motion. In many important machines all the moving points have motion of this kind.[1] In general the motions of any such link are completely determined by the motions of any two points, or of any line, lying in the plane section under consideration. In Fig. 43 let the irregular figure represent any rigid body having plane motion and suppose that the points A and B are constrained to follow the paths m and n respectively. For any chosen position of A there is a perfectly determinate position for B, and from these two the position of any third point C of the system is completely determined. It is therefore sufficient in studying the plane motion of any rigid body to confine the discussion to the motion of any line such as AB,

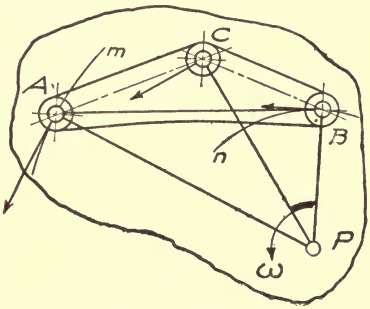


FIG. 43.

19. Rotation about a Fixed Center.—The simplest case of plane motion is that of rotation about a fixed center. In Fig. 44 let PA_1B_1 be the rigid system rotating about the center P. After an interval of time the body has turned through an angle θ and has reached the position PA_2B_2. Evidently the angle $A_1PA_2 =$

[1] In such machines the links may be represented by plane sections or projections on a plane parallel to the planes of motion.

$B_1PB_2=\theta$, and the line A_2B_2 also makes the angle θ with A_1B_1. The displacements of A_1 and B_1 are proportional to the radii PA_1 and PB_1. Measured along the arcs these displacements are $PA_1 \cdot \theta$ and $PB_1 \cdot \theta$ respectively. Measured along the lines A_1A_2 and B_1B_2 the displacements are $2PA_1 \sin \frac{\theta}{2}$ and $2PB_1 \sin \frac{\theta}{2}$, therefore $\frac{\text{displacement of } A_1}{\text{displacement of } B_1}=\frac{PA_1}{PB_1}$.

If the center P is removed to an infinite distance from A_1 and B_1 the ratio $\frac{PA_1}{PB_1}$ approaches 1 as a limit and the angle $\theta=2\sin^{-1}\left(\frac{A_1A_2}{2PA_1}\right)$ approaches 0. If the distance PA_1 becomes infinite the motion becomes a translation. In this case θ equals 0 and $A_1A_2=B_1B_2$, or in other words the displacements of all points of the system are equal and parallel.

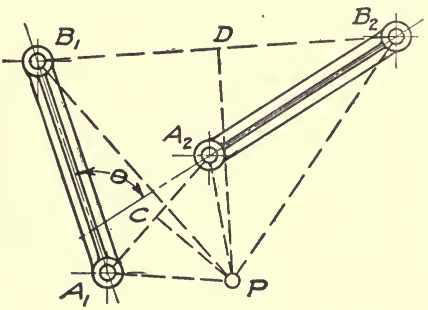


Fig. 44.

20. Any Displacement Equivalent to a Rotation.—Suppose the line A_1B_1, Fig. 44, to be moved to the position A_2B_2 by any path whatever. Join A_1A_2 and B_1B_2 and on these lines erect perpendicular bisectors which intersect at P. Then the displacement from A_1B_1 to A_2B_2 could be effected by a single rotation about P. The truth of this statement is evident from the equality of the triangles PA_1B_1 and PA_2B_2, the three sides of one being equal to three sides of the other.

21. Instantaneous Center.—If the line A_1B_1, Fig. 44, is part of a machine it is generally impossible for the displacement to take place in the form of a simple rotation, for the points A_1 and B_1 will usually not be able to travel in circular arcs about P. Take for example the connecting rod of a steam engine, Fig. 45. If this were a free body it could be moved from the position AB to $A'B'$ by a simple rotation about point P. But the constraints imposed upon the motions of the rod by the crosshead and crank

make such a motion impossible. If, however, the displacement be made extremely small as in Fig. 46, then a very small rotation about P may become possible if there is any lost motion in the joints. In the limit, if the displacement be reduced indefinitely, the point P approaches P_0, the intersection of the normals erected at A and B to the paths of these points. For this case an infinitesimal rotation around P_0 is possible. The motion of the connecting rod may then be considered as composed of a series of infinitesimal rotations about a series of centers P_0. As each point P_0 is the center of rotation only for an instant it is known as an *instantaneous center.*

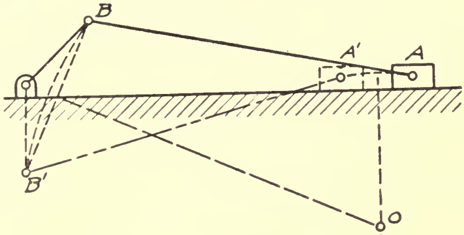


Fig. 45.

What has been said about this particular case applies equally well to any link in any mechanism; the instantaneous motion may be regarded as a rotation about an instantaneous center, and the total motion as a series of infinitesimal rotations about a series of instantaneous centers. In the case of a link rotating about a fixed point the instantaneous center always coincides with the fixed point.

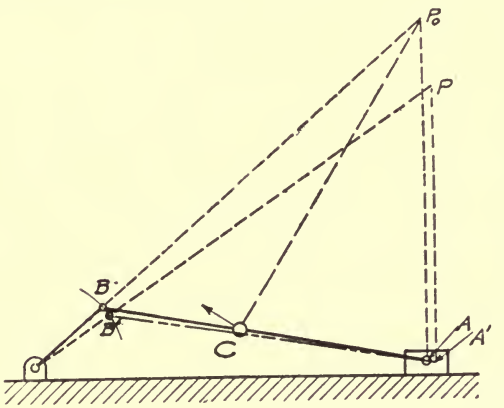


Fig. 46.

22. Properties of the Instantaneous Center.—In the preceding article the instantaneous center P_0 was determined by drawing normals to the directions of the motion of two points on the link, Fig. 45. Now this center may be used in turn to determine the

direction of the instantaneous motion of any other point C on the link. This direction is normal to the instantaneous radius P_0C. It has been shown that the displacements of points on the link are proportional to the instantaneous radii. As the displacements take place in the same element of time dt it follows that the velocities are also proportional to the radii. Expressed mathematically: Vel. A; Vel. B; Vel. $C = P_0A$; P_0B; P_0C.

The size and shape of the link has no effect on this motion so long as the relative positions of the pairing elements are unchanged. We may, therefore, imagine a plane attached to the link and

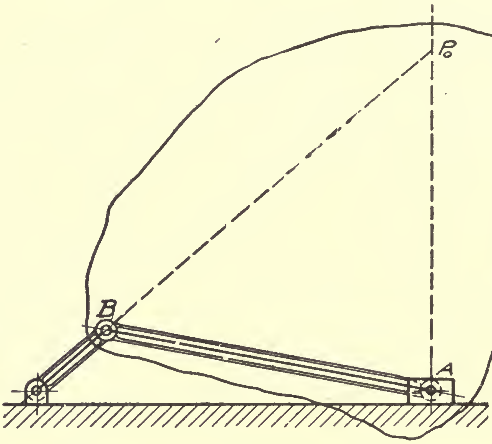


Fig. 47.

moving with it, that is, we may imagine the link expanded into a plane of indefinite extent, but still constrained by the pairs A and B. If this plane is extended to include the instantaneous center P_0 (see Fig. 47) then P_0 is a point on the plane which has no motion. A pin might be stuck through the plane at this point into the fixed link without interfering with the instantaneous motion. The instantaneous center might then be defined as the point on the link (extended if necessary) which for the instant is at rest.

The only possible instantaneous motion of a link in a mechanism is a rotation about the instantaneous center. Hence, any force applied to the link tends to produce such rotation. The

direction of motion is determined by the sense of the moment of the force about the instantaneous center.

23. Combined Motions.—In many cases it is more convenient to regard the motion of any link as the resultant of several motions rather than as a simple rotation about an instantaneous center. For example, the motion of the wheels of a locomotive may be regarded as a translation with the engine together with a rotation about the axles; the motion of the connecting rod of an engine may be regarded as made up of a translation with the crosshead together with a rotation around the wrist pin. Many other examples will occur to the reader.

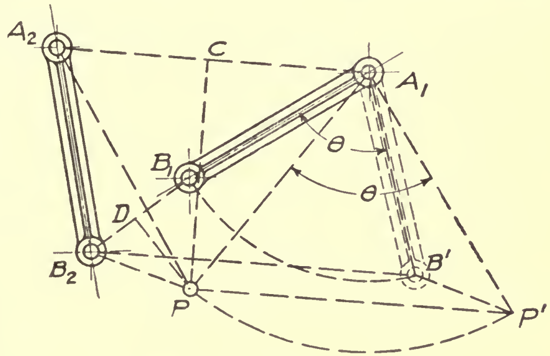


FIG. 48.

Referring again to Fig. 44 it was seen that the displacement from A_1B_1 to A_2B_2 was effected by a simple rotation about P. This displacement, however, could be effected in many other ways. For example, in Fig. 48 the triangle A_1B_1P is first rotated about A_1 through an angle θ. This brings A_1B_1 to the position A_1B', which is parallel to A_2B_2, and shifts P to P'— a distance of $2A_1P \sin \frac{\theta}{2}$. If now the whole triangle $A_1B'P'$ is given a translation $A_1A_2 = B'B_2 = P'P = 2A_1P \sin \frac{\theta}{2}$. A_1 will fall at A_2, B' at B_2 and P' at P. Thus the original rotation around P has been replaced by an equal rotation about A_1 together with a translation $2A_1P \sin \frac{\theta}{2}$. From this result the following principle is deduced: A rotation θ about any center is equivalent to an equal rotation about another center together with a translation equal to twice the distance between the centers times the sine of $\frac{\theta}{2}$. In the case

of an infinitesimal rotation $d\theta$ about an instantaneous center, $2\sin\frac{d\theta}{2}$, becomes equal to $d\theta$, and therefore an infinitesimal rotation $d\theta$ about an instantaneous center is equivalent to an equal rotation about another center together with a translation $\rho d\theta$, where ρ is the distance between the centers. The direction of the translation is normal to the instantaneous radius ρ and is readily seen to be the displacement of the second center.

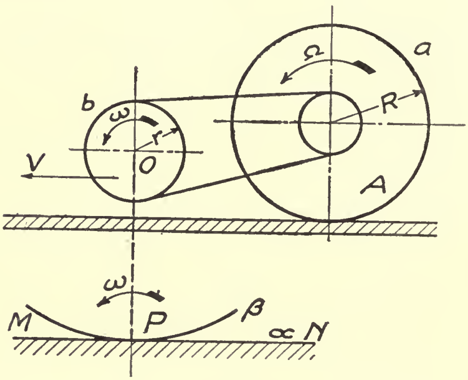


FIG. 49.

As an example consider the motion of the sprocket wheel b, of a bicycle, Fig. 49. The bicycle is moving forward with velocity $V=R\Omega$. The sprocket wheel therefore has a combined motion of translation with velocity V, together with a rotation about its center O with angular velocity ω. Since the chain runs without slip

$$\omega r=\Omega r_1=\frac{V}{R}.r_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total motion of the sprocket wheel is a rotation with angular velocity ω about some instantaneous center P. The velocity of the center O is given by the equation

$$V=OP\omega=OP\frac{Vr_1}{Rr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and therefore

$$OP=R\frac{r}{r_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (3) gives the distance of the instantaneous center P from the center O. Since the instantaneous radius OP must be perpendicular to the motion of O, P lies vertically below O.

The motion of the sprocket wheel can be exactly replaced by the rolling of a wheel of radius OP along a horizontal line MN.

EXERCISES

1. A wheel rolls along the ground as in Fig. 50. Compare the results obtained for the velocity of the point A, (1) regarding the wheel as rotating about P with angular velocity ω. (2) regarding the wheel as having a combined motion of translation with the center O and rotation about O.

2. In Fig. 46 compare the results found for the velocities of B and C (1) regarding the rod as rotating with angular velocity ω about P, (2) regarding the rod as having a combined motion of rotation about A and translation with A.

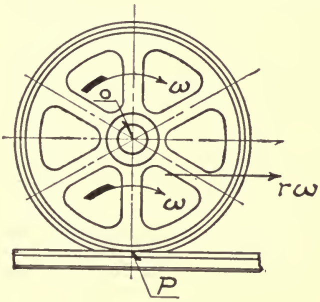


FIG. 50.

3. The wheel Fig. 50 is 6 feet in diameter and rotates at 200 r.p.m. If the center advances at 60 feet per second find the speed of slipping; find the instantaneous center. Compare the velocity of slipping found by the method of instantaneous centers with that found by the method of combined rotation and translation.

24. Instantaneous Center of Relative Motion.—Let the two irregular bodies *2* and *3*, Fig. 51, represent two links of a mechanism. These links may be regarded as two planes of indefinite extent rotating about the instantaneous centers P_2 and P_3. Any point M may be considered as covering two coincident points one on each link. These points may be distinguished as M_2 and M_3. As M_2 is rotating about P_2 with angular velocity ω_2 and M_3 about P_3 with angular velocity ω_3, the velocities of these points will in general be different in both magnitude and direction. If the point is chosen somewhere on the line joining P_2 and P_3 as at A the directions of the velocities will be the same but the magnitudes will in general, be different. There exists, however, one point where the velocities will coincide in both magnitude and direction. This point may be found as follows: From any point A on the line joining P_2P_3 draw vectors representing the velocities of A_2 and A_3. Join the ends of these vectors to P_2 and P_3 respectively. From the intersection of these lines drop a perpendicular on P_2P_3. The foot of this perpendicular B, is a point whose velocity is the same whether considered as a point

on link *2* or *3*. The point B is called the instantaneous center of relative motion. If the mechanism is inverted so that link *2* becomes stationary, B becomes the instantaneous center for link *3*. It is the only point where a pin could be placed connecting links *2* and *3* without interfering with their instantaneous relative motion.

25. Law of Three Centers.—The centers P_2 and P_3, Fig. 51, are the instantaneous centers of motion of the links *2* and *3* relative to the fixed link—say the paper. If we call this link *1*, we see that the centers of relative motion between *1* and *2*, *2* and *3*, and *1* and *3* lie in a straight line. If some other link 4 (not shown) were taken as the stationary link, links *2* and *3* would rotate about two other centers Q_2 and Q_3. These are the centers of

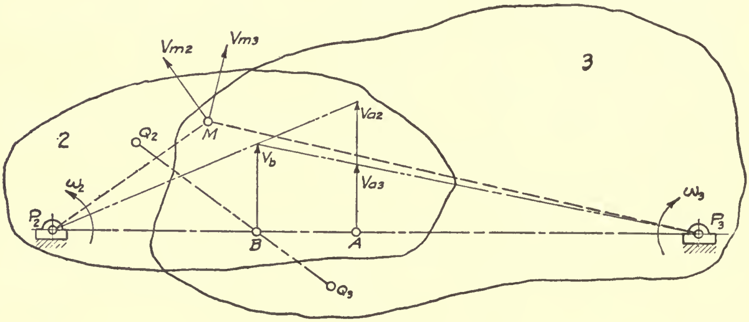


FIG. 51.

relative motion between links *2* and *4*, and *3* and *4*. Reasoning similar to that in the last paragraph will show that the center of relative motion between *2* and *3* will lie on the line Q_2 Q_3. In general, if three links r, s, and t are chosen, *the centers of relative motion between r and s, r and t, and s and t, lie in a straight line.* This law is called the *law* of *three centers.* For the sake of brevity instantaneous centers of relative motion will be denoted by the symbols *12*, *16*, *28*, etc., where *12* represents the center of relative motion between links *1* and *2*, etc. These symbols should be read one-two, one-six, two-eight, etc., not twelve, sixteen, twenty-eight. The law of three centers is of great importance in the theory of machines and is used extensively in studying machine motions.

26. Application of the Law of Three Centers.—It is usually possible to find all of the instantaneous centers of relative motion in a mechanism by the law of three centers. As the simplest case, consider the 4-link chain, Fig. 52. Here are four systems all having motions relative to one another. In all, there are six relative motions and consequently six centers of relative motion, namely: *12, 13, 14, 23, 24, 34*. Of these six relative motions, four are rotations about the axes of the joints namely: *12, 23, 34, 14*; and of the six centers, four are located at the axes of the joints as shown. The remaining centers, *13* and *24* are found by the law of three centers. Thus, considering the links *1, 2* and *3*, the centers *12, 23* and *13* must lie in a straight line. As the centers *12* and *23* are known, this determines a locus for *13*. Again considering links *1, 3* and *4*, the centers *14, 34* and *13* must lie in a straight line. This determines a second locus for *13*, which must, therefore, lie at *M*, the intersection of the loci. Similarly, *24* must lie on the lines *12–14* and *23–34* and is consequently located at *N*, the intersection of these lines. It is evident that if *2* is held stationary, *N* is the instantaneous center of rotation of *4*, and that if *3* is held stationary, *13* is the center around which *1* rotates.

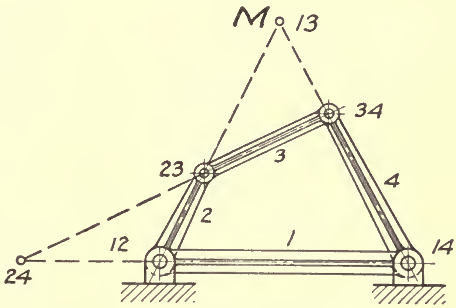


Fig. 52.

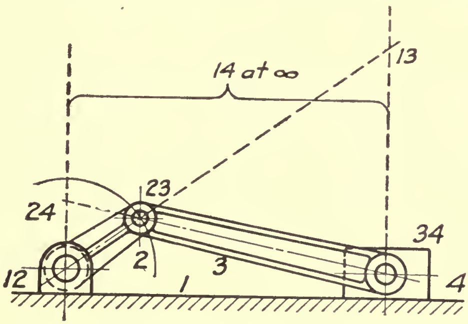


Fig. 53.

In Fig. 53 is shown the slider crank or steam engine mechanism. In this case, since link 4 has a motion of translation, the center *14*

lies at an infinite distance. To connect any point to *14*, simply draw a vertical line through that point. The centers *12*, *23* and *14* are located at the axes of the turning joints. To find the unknown centers, *13* and *24*, use the law of three centers. *13* lies on the lines *12–23*, and *14–43*. *24* lies on the lines *12–14* and *23–34*. This may be conveniently represented as follows:

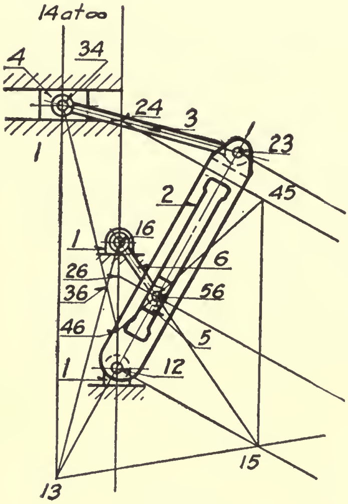


FIG. 54.

13	12–23 14–34	24	12–14 23–34

As an example of the application to more complicated mechanisms, consider the shaper mechanism, Fig. 54. In general, if a mechanism has n links, there are $\frac{n(n-1)}{2}$ centers of relative motion. Thus, in the six-link mechanism there are $\frac{6(6-1)}{2} = 15$ centers. These may be conveniently tabulated as shown below:

12	13	14	15	16
	23	24	25	26
		34	35	36
			45	46
				56

In this table all known centers may be indicated by small full circles. An examination of the mechanism shows the following known permanent centers: *12*, *14*, *16*, *23*, *25*, *34*, *56*, and these are accordingly enclosed in full circles in the table. The remaining 8 centers are to be determined by the law of three centers. Further examination shows two *quadric* or four-link chains in the

mechanism, namely *1234* and *1256*. The centers *13*, *24*, *15* and *26* are then readily found by the same method as that employed in Fig. 53. These are now marked in the table by dotted circles. The remaining centers are found as follows:

35	13–15 23–25	12–25 16–65	
13	12–23 14–43		
36	16–13 23–26	12–16 25–56	
45	14–15 24–25		
24	12–14 23–34		
46	14–16 24–26 34–36		These lines coincide, so a third locus must be found.

As checks on the accuracy of the work, it should be noted that *23*, *26*, *36*, must lie in a straight line; *34*, *35*, *45* must lie in a straight line; *35*, *36*, *56*, must lie in a straight line; *45*, *46*, *56* must lie in a straight line.

27. General Directions.—The following general directions will be found useful:

1. Construct a table such as shown in the preceding article including all the centers.
2. Indicate all known centers by drawing full circles around them. Note that all axes of turning joints are centers, and that the centers of sliding pairs are at an infinite distance in a direction perpendicular to the direction of relative motion.
3. Examine the mechanism and pick out any quadric

chains. If necessary, draw the skeleton in its typical form to assist in picking out these chains. As in Figs. 52 and 53 intersections of opposite sides of quadric chains give additional centers which may be marked by dotted circles.

4. Apply the law of three centers to find any center which is still unknown.
5. Test the accuracy of the work by drawing all loci which pass through each center. If there are n links, then there are $n-2$ loci for each center, e.g., the center *12*

$$\text{lies on the lines}\ \left|\begin{array}{l} 13\text{–}23 \\ 14\text{–}24 \\ 15\text{–}25 \\ \cdot\ \cdot\ \cdot \\ 1n\text{–}2n \end{array}\right.$$

If one of these lines, say *15–25*, has not been drawn, see whether it passes through *12*. The total number of loci [1] is $\frac{n(n-1)(n-2)}{1.2.3}$.

28. Higher Pairs.—To determine all the instantaneous centers of mechanisms with higher pairs, it is usually necessary to draw normals to paths of points or to contact surfaces. One or two examples will illustrate this statement.

In Fig. 55 two members, *2* and *3*, rotate about fixed points at *12* and *13*, respectively. Link *3* drives link *2* by direct contact, thus giving a case of higher pairing. It is required to determine the center *23* of the relative motion of links *2* and *3*. Since links *2* and *3* cannot cut into each other, and since they do not separate, the relative motion of the points in contact must be along the common tangent t at the point of contact P; for if the relative

[1] This is not as long and tedious a process as it may appear. Thus, for the six-link mechanism, Fig. 54, there are 20 loci as follows: Those not drawn on the figure are indicated by stars. Such loci are useful as checks on the accuracy of the work.

12	13–32, 14–42, 15–52, 16–62
13	14–43, 15–53, 16–63
14	15–54, 16–64
15	16–65
23	24–43, 25–53, 26–63*
24	25–45, 26–46
25	26–56
34	35–54*, 36–64
35	36–56*
45	46–56*

motion has a component along the normal, the members must either cut into each other or separate. But the motion of *2* relative to *3* is a rotation about the center *23* and evidently, therefore, this center must lie on a line perpendicular to the direction of the rela-

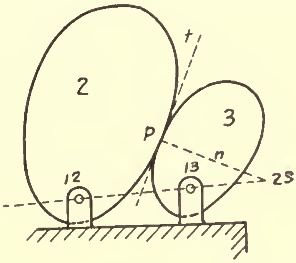


Fig. 55.

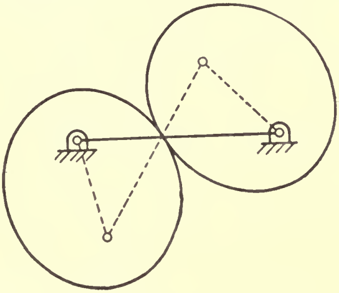


Fig. 56.

tive motion, that is, perpendicular to the tangent t; hence *23* lies on the common normal drawn through P. By the law of three centers, *23* also lies on the line *12–13*; hence it lies at the intersection of the latter line with n.

In the case of parts rolling upon each other as the pitch lines of circular or elliptical gears, Fig. 56, the point of contact must lie on the line of centers and is itself the instantaneous center of the two systems.

In Fig. 57 there is a higher pair between the pin on the block *2* and the slot in the swinging arm *3*. The relative motion of the pin and slot is a sliding along the slot, and the center *23* lies therefore in a line through the pin perpendicular to the direction of the center line of the slot. The center *12* is at infinity. The locus *12–13* is therefore a line through *13* perpendicular to the motion of link *2*. By the law of three centers *23* lies also on the line *12–13* and is therefore fully determined.

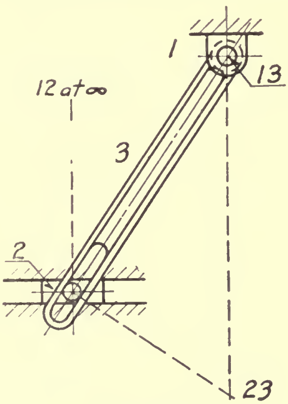


Fig. 57.

29. Special Cases.—In some cases the law of three centers alone is insufficient to determine all the instantaneous centers. Thus in the skeleton mechanism, Fig. 58 (which is the same as Fig. 28), the centers *13* and *24* are readily found; but it may be shown by trial that none of the remaining centers can be determined by the law of three centers. Another center may be found by a sort of trial and error method as follows:

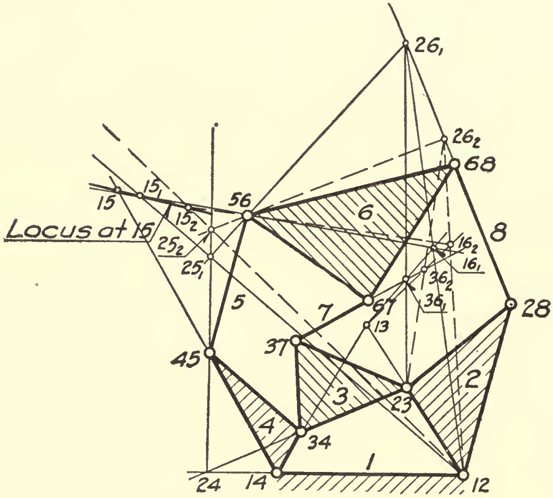


Fig. 58.

The center *26* is known to lie on the line *28–68*. Assume this center to be at some point on this line as 26_1. Then other centers may be determined as follows:

$$15_1 \left| \begin{array}{l} 12 - 25_1 \left| \begin{array}{l} 24 - 45 \\ 26_1 - 65 \end{array} \right. \\ \\ 65 - 16_1 \left| \begin{array}{l} 12 - 26_1 \\ 13 - 36_1 \left| \begin{array}{l} 23 - 26_1 \\ 37 - 76 \end{array} \right. \end{array} \right. \end{array} \right.$$

A trial position of center *15* is thus located. But *15* must lie on the line *14–45*. Hence the assumption as to the location of center *26* at 26_1 is incorrect. Assume center *26* to lie at 26_2 and repeat

the process as shown. By a series of such assumptions a number of trial positions of *15* are found. These determine a locus for *15*, and the point where this locus cuts the line *14–45* is the true position of *15*. After *15* has been located correctly the remaining centers can be found by the law of three centers.[1]

30. Centrodes—If any link, say link *5*, of a machine has motion relative to the fixed link *1*, then this motion may be regarded as a series of rotations around successive instantaneous centers. If a curve be drawn connecting all the positions of the instantaneous center this curve is called the *fixed centrode.* This locus may be regarded as a curve attached to link *1* and always containing the center *15*. Similarly, if a plane be attached so as to move

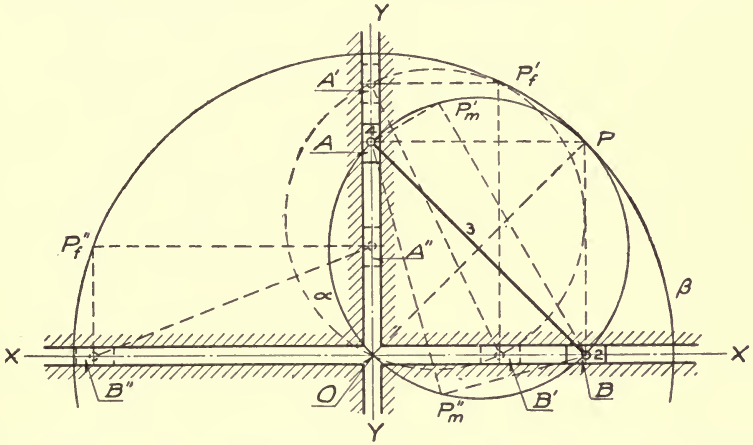


FIG. 59.

with link *5*, a second curve which always contains the center 15 can be drawn on this plane. The second curve is called the *moving centrode.* For example, if a wheel rolls along a rail, the instantaneous center of the wheel is always at the point of contact with the rail. The top of the rail is, therefore, the locus of this center, that is, the fixed centrode. The rim of the wheel also always contains the instantaneous center and is, therefore, the moving centrode. A second example is shown in Fig. 59. Two points *A* and *B* of a rod are caused to move along two grooves

[1] When this method is followed the locus is usually found to be a straight line. In some cases, however, conics or even higher degree curves are involved.

which intersect at right angles. The instantaneous center 13 is readily located at P. As the distance OP is always equal to AB, it follows that the fixed centrode is a circle about O with radius OP equals AB. The moving centrode is easily found to be a circle whose diameter is AB. For the angle APB is always a right angle, and the locus of a point at which a line AB subtends a right angle is a circle whose diameter is AB. Thus if the bar moves (carrying the circle APB with it) to the position $A'B'$, the new center P' lies on this circle.

In most cases the geometrical character of the centrodes is not easily determined. They must then be constructed point by point. A convenient way to construct the moving centrode is as follows: On a sheet of tracing paper, make a drawing of the moving link, as AB. Place this tracing in the successive positions of the link as AB, $A'B'$, etc., and at each position mark on the tracing paper the corresponding instantaneous center. The points thus located on the tracing paper determine the moving centrode.

31. Centrodes of Relative Motion.—The relative motion of any two links may be considered as a series of rotations about successive instantaneous centers of relative motion. A pair of curves, one attached to and moving with each link, and which always contain this center, constitute the centrodes of relative motion of these links. Each of these centrodes is readily found by means of a sheet of tracing paper as previously described. There is, therefore, a pair of centrodes for each pair of links in a chain, or if there are n links, $\frac{n(n-1)}{2}$ pairs in all. In case two links are connected by a turning joint, both centrodes reduce to a point—the axis of the joint.

EXERCISES

4. In Fig. 60 let *1* be the fixed link and *2* and *4* the cranks rotating about P_2 and P_4 respectively. Taking link *3* as a moving system S, construct the fixed and moving centrodes.

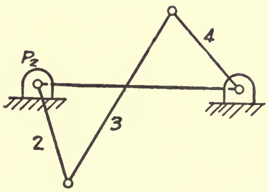


Fig. 60.

Make $P_2A = P_4B = 3''$ and $P_2P_4 = AB = 2''$.

Construct centrodes of relative motion between links 2 and 4.

5. The moving system S, Fig. 61, consists of two lines rigidly connected and making an angle of 60° with each other. The motion of

the system is such that one of these lines always passes through a fixed point M, the other through a fixed point N. Construct the centrodes. Make the distance $MN = 4''$.

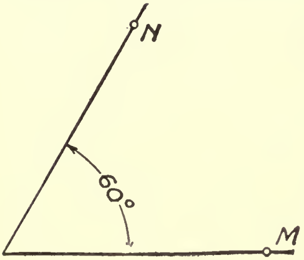


FIG. 61.

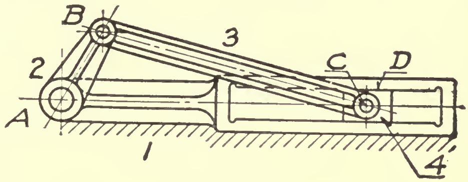


FIG. 62.

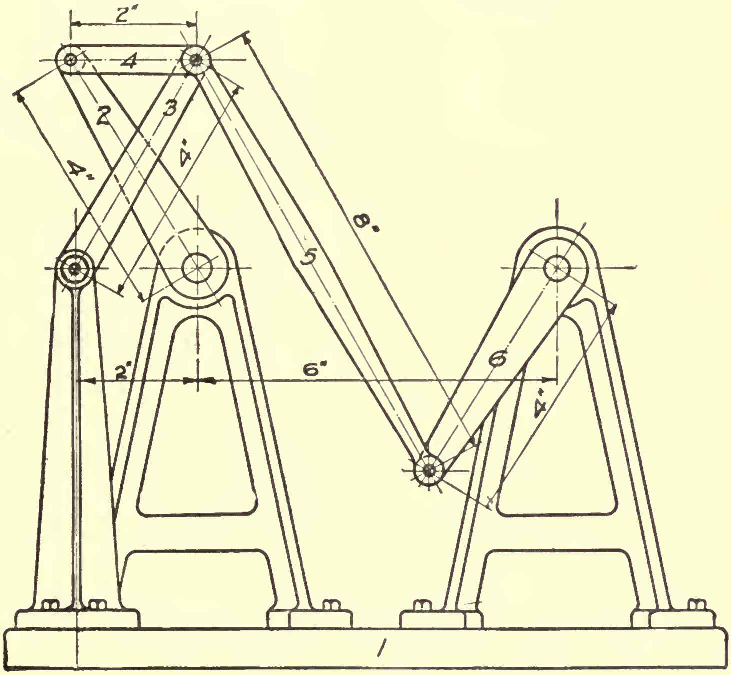


FIG. 63.

6. In the slider crank mechanism, Fig. 62, take *1* as the fixed link, and *3* as the moving system S, and construct the centrodes attached to *1* and *3*, respectively. Let $AB = 1''$ and $BC = 3\frac{1}{2}''$.

7. In the same mechanism take *4* as the fixed link and construct the centrodes attached to *4* and *2*, respectively.

8. In the six-link mechansim, Fig. 63, construct the centrodes attached to links *3* and *6*, respectively.

32. Motion Produced by Rolling of Centrodes.—At the point of contact between two centrodes, that is, the instantaneous center of relative motion, there can be no relative motion between the two links to which the centrodes belong. Therefore the centrodes roll on each other without slip. Consequently the motion of any links can be reproduced by simply rolling the two centrodes together. For example, in Fig. 59 if the bar AB be detached from the guides, links *2* and *4*, its motion can be reproduced by simply rolling the smaller circle on the inside of the larger. This is accomplished in the mechanism shown in Fig. 41. Here the large internal gear 1 is held stationary and the smaller gear link *3* is caused to roll inside *1* by means of the crank link *2*. It is easily proved that two points, A and B, on the pitch circle at opposite ends of a diameter travel in straight lines which are perpendicular to each other. The bar *4* adds nothing to the constraint. It simply emphasizes the fact that B must travel along a vertical diameter of the larger gear.

33. Equivalent Mechanisms.—Since the motions of the links may be completely replaced by the rolling of the centrodes on each other, it follows that mechanisms which give the same centrodes are kinematically equivalent. This equivalency can be shown to exist in machines bearing no external resemblance to each other. Thus the elliptograph, Fig. 59, the Davis engine, Fig. 34, the hypocyclic parallel motion, Fig. 41, and the slider crank or steam-engine mechanism where the crank and connecting rod are of the same length, all give the same centrodes between links *1* and *3*. These mechanisms therefore are all equivalent to each other, and suitably chosen points on each may be shown to follow identical paths.

34. Gears as Centrodes.—In geared mechanisms, the pitch lines of the spur gears (whether circular or not) roll on each other without slip. These pitch lines are therefore centrodes. In the crossed four-bar chain, Fig. 60, for example, the centrodes between links *1* and *3* are found to be ellipses. The whole mechanism might then be replaced by a pair of elliptical gears each rotating about a focus of the ellipse. In any mechanism the motions of any two links may be reproduced by substituting for these links a pair of gears whose pitch lines are the centrodes of relative motion of these links. Of course in many cases this is practically impos-

sible; the centrodes may be of infinite length, or of shapes quite unsuited for gears; but theoretically this substitution is always possible.

35. Pitch Lines.—The determination of pitch lines of gears may be approached from two points of view. If the gears are to replace some linkage whose properties are known then the pitch lines must be the centrodes as shown in the preceding paragraph. If on the other hand the pitch line of one gear is arbitrarily chosen a second gear can be designed so as to insure proper meshing with the first. In Fig. 64 let the closed curve *2* represent the arbitrarily chosen pitch line of one gear, which is to rotate about the center P_2. It is required to find the pitch line of a second gear which will rotate about P_3 and mesh with the first. As the pitch curves are to roll on each other, the instantaneous center of relative motion is at the point of contact, which therefore by the law of three centers must lie on the line P_2P_3. It follows that the sum of the radii to the point of contact must be constant. Divide the perimeter of gear *2* into parts so small that the length of each chord may be considered equal to its arc. From A strike an arc of radius AB, and from P_3 an arc of radius $P_2P_3-P_2B$. The intersection of these arcs gives the point B' of the second gear, which is to come in contact with B. From B' strike an arc of radius BC and from P_3 an arc of radius $P_2P_3-P_2C$. The intersection of these gives C'. This process can be continued until the profile of the second gear is completed.[1]

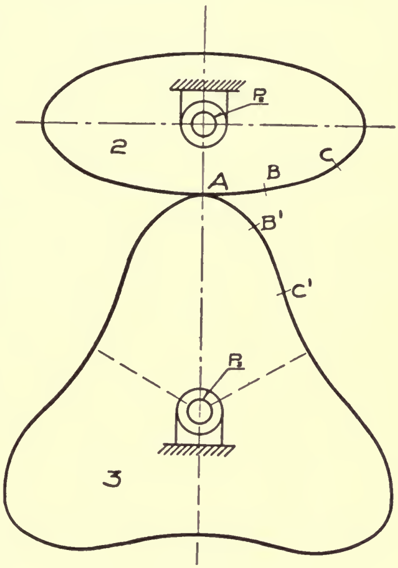


FIG. 64.

Certain special forms of pitch lines such as ellipses and log-

[1] Ordinarily two non-circular gears will have the same perimeter, or

arithmic spirals are found to be suitable for non-circular gears.[1] Of course in the case of spirals the curves must have points of discontinuity.

36. Gear Teeth.—The actual contact between gear teeth is in general not at the pitch point but at one side. Consequently the actual pairing between two toothed wheels is of the same nature as that described for cams. In order that the motion of the wheels shall be actually equivalent to the rolling of the two pitch curves on one another, the instantaneous center of relative motion must lie at the pitch point. But this center lies on the common normal at the point of contact. Consequently for correct action the teeth must be cut in such form as to satisfy the condition that the common normal always passes through the pitch point. Theoretically if the profile of the teeth of one gear is arbitrarily chosen, a profile can be found for the second so as to give correct motion.[2] In practice only two forms are used extensively, namely the *involute* and *cycloidal.* The discussion of the form and construction of tooth profiles will be found in Chapter X.

37. Space Motion. Axodes.—In all the preceding work it was assumed that every point of the mechanism had plane motion only. In the more general case, the points may travel in any paths in space whatever. The most general instantaneous motion of a rigid body in space may be shown to be equivalent to an infinitesimal rotation about some axis together with a translation along that axis. For example, a nut moving on a stationary bolt rotates about the axis and at the same time travels along the axis. The most general motion possible may be resolved into a series of such infinitesimal rotations and translations.

If a surface be passed through all the positions of the instan-

their perimeters will have a simple ratio such as *2 : 1*, *3 : 1*, etc. It should be noted that in general the process described will not give closed pitch curves which satisfy this condition. That is, the gears cannot make complete revolutions. The conditions which must be satisfied in order that the second gear shall make one revolution while the first makes *1*, *2*, ...n, revolutions, are discussed in note (A). The application of these principles is a matter of considerable difficulty. The converse problem of making the second gear revolve *1*, *2*, ...n, times for each revolution of the first, is in general not capable of solution.

[1] Dunkerly, Mechanism, pp. 339–346.

[2] Dunkerly, Mechanism, pp. 275–280 and 353–356.

taneous axis, this surface is called an axode. Exactly as in the case of centrodes, there exists for any constrained motion of a rigid body a fixed axode and a moving axode. The moving axode is a second ruled surface attached to the body and moving with it and which always contains the instantaneous axis of rotation. The motion of the body may be exactly reproduced by rolling the moving axode on the fixed, and at the same time sliding it along the element of contact. Also the relative motions of two moving bodies may be replaced by the rolling and sliding of two axodes.[1]

The only important applications of axodes are in conical and hyperboloidal (skew bevel) gears. In conical gears the axes of rotation intersect, and the motion is exactly equivalent to the rolling of the two pitch cones on each other. The axodes are these pitch cones. In this case there is no slip in the direction, of the line of contact. In skew bevel gears the axodes are the hyperboloidal pitch surfaces and here both rolling and sliding occur.

[1] For further discussion of the question of axodes the reader is referred to Weisbach's Mechanics of Engineering and Machinery, Vol. 2, introduction.

CHAPTER III

VELOCITIES IN MECHANISMS

38. Introductory.—In the preceding chapters the motions of machine parts have been considered without particular reference to the velocities or rates of motion. In the present chapter methods will be developed by means of which the velocities in all parts of a machine can be completely determined. These methods may be classified as *Analytical* and *Graphical*. Of the latter there are two general classes—namely, the *method of Instantaneous Centers*, and the *method of Relative Velocities*. In some cases a combination of these two may be profitably employed.

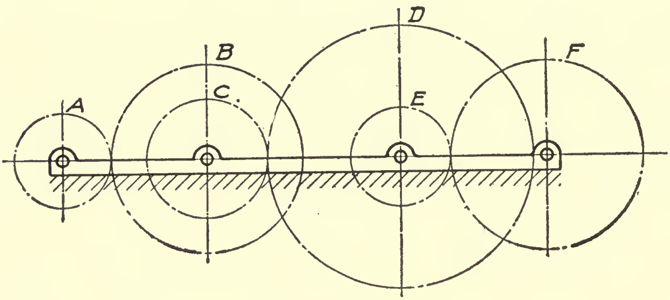


Fig. 65.

39. Analytical Methods.—Analytical methods are used extensively in geared mechanisms and in some very simple linkages. For more complex linkages the mathematical difficulties become so great and the equations so cumbersome that graphical solutions are much to be preferred.

(*a*) The most useful field for the employment of analytical methods in finding velocities is in the case of geared mechanisms. Consider the ordinary gear train, Fig. 65. Let a, b, c, d, e, and f

represent the numbers of the teeth in the gears A, B, C, D, E, and F respectively. It is required to find the ratio of the angular velocity of gear F to that of gear A. The numbers of teeth in two gears which are in mesh are proportional to the circumferences of the pitch circles. As the pitch circles roll on each other without slip, evidently the number of revolutions of the gears are inversely proportional to the circumferences of the pitch circles, or to the numbers of teeth.

Therefore:

$$\frac{\omega_b(=\text{angular velocity of } B)}{\omega_a(=\text{angular velocity of } A)}=-\frac{a}{b},$$

or

$$\omega_b=-\omega_a\frac{a}{b}.$$

If B and C are rigidly connected so that they must rotate together

$$\omega_c=\omega_b; \quad \text{similarly} \quad \omega_d=\omega_e=\omega_c\left(\frac{c}{d}\right)=\omega_a\left(\frac{a}{b}\cdot\frac{c}{d}\right),$$

and

$$\omega_f=\omega_e\left(\frac{e}{f}\right)=\omega_a\left(\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}\right),$$

hence

$$\frac{\omega_f}{\omega_a}=\frac{ace}{bdf}, \text{ etc.}$$

From this analysis we deduce this important law of gearing: The angular velocity of the driven wheel F is to the angular velocity of the driver A as the continued product of the numbers of teeth in the driving wheels (A, C, E) is to the continued product of the numbers of teeth on the driven wheels (B, D, F). Attention must be paid to the direction of rotation. Thus if A rotate in the counter-clockwise (positive) direction B and C will rotate in the clockwise (negative) direction, D and E counter-clockwise and F clockwise. The angular velocities are completely described therefore in the following table:

Gear	A	B	C	D	E	F
Angular veolcity.	ω_a	$-\omega_a\left(\frac{a}{b}\right)$	$-\omega_a\left(\frac{a}{b}\right)$	$+\omega_a\left(\frac{ac}{bd}\right)$	$+\omega_a\left(\frac{ac}{bd}\right)$	$-\omega_a\left(\frac{ace}{bdf}\right)$

(*b*) In many cases the same gear appears both as driver and driven. Such a gear is called an idler, and is shown in *B*, Fig. 66. For this case

$$\omega_b = -\omega_a\left(\frac{a}{b}\right),$$

$$\omega_c = +\omega_a\left(\frac{ab}{bc}\right) = +\omega_a\left(\frac{a}{c}\right).$$

It follows that an idle gear does not change the magnitude of the velocity ratio, but does alter the direction or sign.

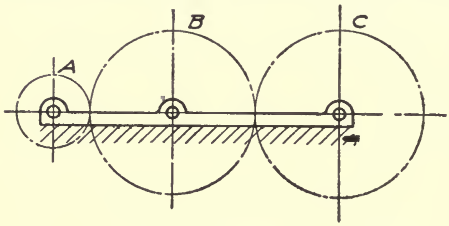


FIG. 66.

(*c*) In many cases the pitch circles of two gears are tangent internally instead of externally. In these cases the larger gears are called internal gears. Thus in Fig. 67, *A* is an internal gear. The velocity ratio is found as before:

$$\frac{\omega_b}{\omega_a} = \frac{a}{b}.$$

It should be noted, however, that in this case the gears rotate in the same direction.

(*d*) In the foregoing examples it has been assumed that each gear rotates about a fixed axis passing through its center. In many mechanisms one gear is held stationary and a second one rolls round its circumference. Such an arrangement is shown in Fig. 68. The gear *A* link *1* is held fixed and gear *C*, link *3*, rolls on *A*, the arm *B*, link *2*, serving to hold the two in mesh. In this arrangement *C* is called an epicyclic or planetary gear. The motion of *C* may be regarded as composed of a rotation with *B* about the center *12* together with a rotation relative to *B* about the center *23*. Taking first the motions relative to *B*, if *A* be given $+1$ revolution. *C* will make $-\frac{a}{c}$ revolutions relative to *B*. Now the whole mechanism may be given such a motion with *B* that *A* is brought back to its original position. That is, the whole mechanism must be given one revolution in the negative

direction. The total motion of each link is then given by the following table:

A	C	B
$+1-1=0$	$-\frac{a}{c}-1=\frac{-a-c}{c}$	$0-1=-1$

It follows that C must rotate in the same direction as B and that:

$$\frac{\omega_c}{\omega_b}=\frac{a+c}{c}.$$

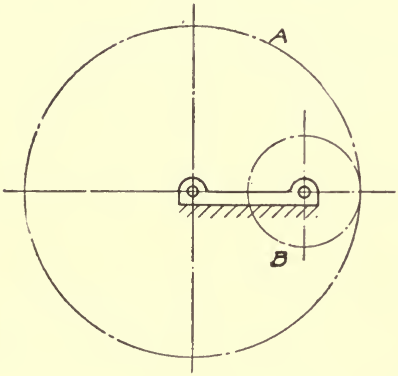


FIG 67.

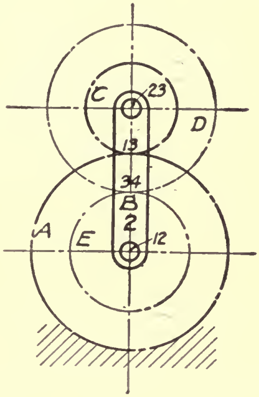


FIG. 68.

(*e*) The same scheme can be followed if we add other gears such as D and E shown dotted in Fig. 68. If D be made to revolve with C, and E be free to revolve about *12*, the motions of each are shown in the following table:

	A	B	C	D	E
Motions relative to B........	1	-0	$-\frac{a}{c}$	$-\frac{a}{c}$	$+\frac{ad}{ce}$
Motions with B.............	-1	-1	-1	-1	-1
Total motion...............	0	-1	$-\left(1+\frac{a}{c}\right)$	$-\left(1+\frac{a}{c}\right)$	$-1+\frac{ad}{ce}$

The gear E can, therefore, rotate either in the same direction as B or the opposite direction, according as $\frac{ad}{ce}<1$ or >1.

Instead of the numbers of teeth the diameters might be used. If the diameter of $A>$ the diameter of E, then $\frac{a}{e}>1$; also $\frac{d}{c}>1$.

Therefore $1-\frac{ad}{ce}<0$ and therefore, the rotation of E is in the opposite direction to that of B, and *vice versa.*

Another conclusion to be drawn is that, since $\frac{ad}{ce}>0$ in all cases, $\frac{\omega_e}{\omega_b}<1$, when the rotations are in the same direction. Since there is no limit to the ratio $\frac{ad}{ce}$, it follows that the ratio $\frac{\omega_e}{\omega_b}$ can be made as large as desired when the rotations are in opposite directions. It also follows that if A and E are of nearly the same diameter the ratio $\frac{\omega_e}{\omega_b}$ becomes exceedingly small. For example, if the numbers of teeth are chosen $a=51$, $c=50$, $d=49$, $e=50$, then,

$$\frac{\omega_e}{\omega_b}=1-\frac{ad}{ce}=1-\frac{2499}{2500}=\frac{1}{2500}.$$

This is a useful principle when very large reductions of speed are desired without the use of an excessive number of gears or very large gears.

(*f*) The same principles may be applied to hypocyclic gear trains, that is, gear trains having a stationary internal gear.[1] The development of the details is left to the reader.

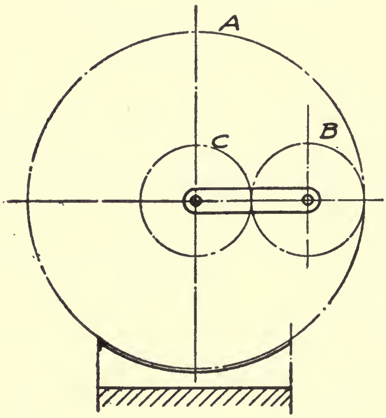


Fig. 69.

EXERCISE

In Fig. 69, gear A has 72 teeth, and B has 32 teeth. Find the number of revolutions of gears B and C for 1 revolution of arm D.

[1] All of the results obtained thus far can be found by the graphical methods described later in the chapter.

40. Bevel-gear Trains.—(*a*) Bevel-gear trains may be treated in a manner similar to that used for spur gears. As before, the ratio of the angular velocities is given by:

$$\frac{\text{Continued product of numbers of teeth in driving wheels}}{\text{Continued product of numbers of teeth in driven wheels}}.$$

The question of direction must, however, be treated differently. Fig. 70 shows an ordinary bevel gear train. As before:

$$\frac{\omega_b}{\omega_a}=\frac{a}{b}.$$

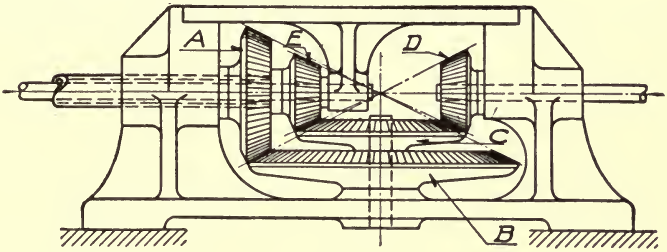


FIG. 70.

But as B and A do not rotate in parallel planes we cannot say that ω_b is either in the same direction as that of A, or in the opposite direction. Consequently, neither the + nor − sign is appropriate to describe the direction.

$$\frac{\omega_d}{\omega_a}=-\frac{ac}{bd}.$$

$$\frac{\omega_e}{\omega_a}=+\frac{ac}{be}.$$

Formal rules can be drawn up for determining the signs, but it is better to consider each case as it arises.

(*b*) Mechanisms involving the use of stationary bevel gears are sometimes employed. The velocity ratios are determined in a manner analogous to that employed with epicyclic spur gear trains. As an example, consider Humpage's gear, Fig. 71. Gear A is stationary. Gears B and C rotate together about a shaft H which is carried by a bracket G, the latter in turn being free

to rotate about the central shaft F. Gear B meshes with the driving gear E and C meshes with D. To determine the velocity ratio $\frac{\omega_d}{\omega_e}$, consider the motions of all gears as composed of rotations

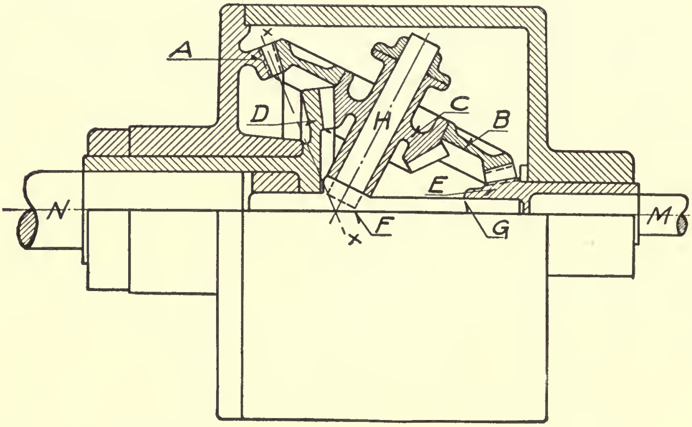


FIG. 71.

relative to H and a rotation with H around the central axis. The following table shows the results:

	A	B	C	D	E	H
Motion relative to H. .	$+1$	$\frac{a}{b}$	$\frac{a}{b}$	$+\frac{ac}{bd}$	$-\frac{a}{e}$	0
Motion with H.	-1	-1	-1	-1	-1	-1
Total motion.	0	$-1,\ \frac{a}{b}$ [1]	$-1,\ \frac{a}{b}$	$-\left(1-\frac{ac}{bd}\right)$	$-\left(1+\frac{a}{e}\right)$	-1

[1] The motion of B and C consists of two rotations about different axes: -1 about the central axis F, and a/b about H. These cannot be added algebraically. They can, however, be combined vectorially into a single rotation about XX, whose magnitude can be found by the ordinary method of combining vectors. The pitch cones of A and B are in short the fixed and moving axodes for the motions of B and C. Fig. 72.

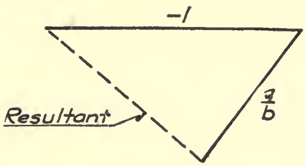


FIG. 72.

Therefore:

$$\frac{\omega_d}{\omega_e}=\frac{1-\dfrac{ac}{bd}}{1+\dfrac{a}{e}}.$$

41. Velocity of Piston of Engine.—In some very simple linkages analytical methods may be used for determining velocities. As an example, consider the slider crank or steam engine mechanism. Here, the crank usually rotates at a nearly uniform speed and the problem is to find the speed of the piston. Referring to Fig. 73, the velocity of the piston is expressed by $\frac{dS}{dt}$.

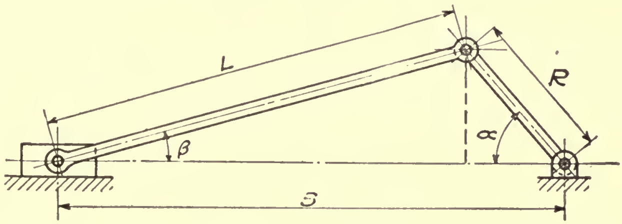


Fig. 73.

From the geometry of the figure

$$S=R\cos\alpha+L\cos\beta.$$

Also $$L\sin\beta=R\sin\alpha.$$

Hence:

$$L\cos\beta=L\sqrt{1-\frac{R^2}{L^2}\sin^2\alpha}.$$

Since $\left(\frac{R}{L}\right)^2\sin^2\alpha$ is a small term we may write with sufficient accuracy,

$$\cos\beta=\sqrt{1-\frac{R^2}{L^2}\sin^2\alpha}=1-\frac{R^2}{2L^2}\sin^2\alpha.$$

This is equivalent to adding the small quantity $\frac{R^4}{4L^4}\sin^4\alpha$ under the radical.

The same result can be obtained by expanding

$$\left(1-\frac{R^2}{L^2}\sin^2\alpha\right)^{\frac{1}{2}},$$

according to the binomial theroem and neglecting all except the first two terms. Substituting this value for $\cos\beta$ a new expression for S is found:

$$S=R\cos\alpha-\frac{R^2}{2L}\sin^2\alpha+L.$$

Differentiating with respect to the time,

$$\frac{dS}{dt}=\left(-R\sin\alpha-\frac{R^2}{2L}2\sin\alpha\cos\alpha\right)\frac{d\alpha}{dt}=-R\left(\sin\alpha+\frac{R}{2L}\sin 2\alpha\right)\omega.$$

It may be remarked that this velocity is much more easily found by graphic methods.

42. Methods of Instantaneous Centers. Angular Velocity Ratios.—Consider any two moving links S and T, Fig. 74. They

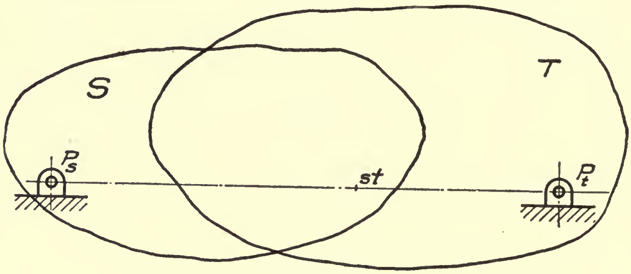


Fig. 74.

are rotating about some instantaneous centers P_s and P_t, with angular velocities ω_s and ω_t respectively. The instantaneous center of relative motion st lies on the line P_sP_t. The center st is defined as a point which has the same velocity whether considered on link S or link T. Considered as a point on link S the velocity of st is given by the equation:

$$V_{st}=(P_s-st)\omega_s.$$

Considered as a point on link T,

$$V_{st}=(P_t-st)\omega_t.$$

Therefore

$$\frac{\omega_t}{\omega_s}=\frac{P_s-st}{P_t-st}.$$

The angular velocities are measured relative to a third link R (say the paper) which is regarded as fixed. The centers P_s and P_t may, therefore, be denoted by rs and rt respectively. In general, therefore:

$$\frac{\omega_{tr}}{\omega_{sr}}=\frac{rs-st}{rt-st}.$$

This equation is to be read as follows:

"The angular velocity of link T relative to link R is to the angular velocity of link S relative to link R as the distance between the centers rs and st is to the distance between the centers rt and st."

If the center st lies between rs and rt, then ω_{sr} and ω_{tr} are in opposite directions. If st lies beyond either of the centers rs or rt, then the two angular velocities are in the same direction.

43. Examples.—(*a*) In two gears revolving about fixed axes the instantaneous center of relative motion is the pitch point. The rule just derived then states that the angular velocities of the gears are inversely proportional to the radii of the pitch circles.

(*b*) In the epicyclic gear train, Fig. 68,

$$\frac{\omega_c}{\omega_b}=\frac{12-23}{13-23}=\frac{R_a+R_c}{R_c}.$$

Compare this result with that obtained analytically in Art. 39.

(*c*) In Fig. 75 link *2* drives link *3* by direct contact. The instantaneous center *23* is at the point where the common normal intersects the line of centers *12 –13*.

$$\frac{\omega_{31}}{\omega_{21}}=\frac{12-23}{13-23}.$$

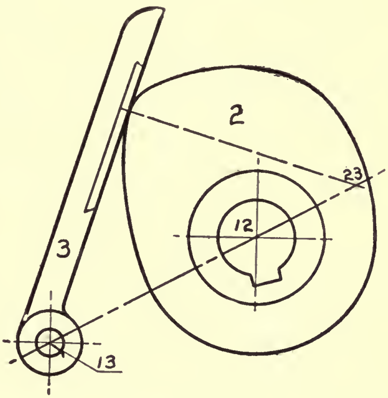


FIG. 75.

That is, the angular velocities of the two links are inversely proportional to the segments into which the common normal divides the line of centers *12–13*.

(*d*) In the steam engine mechanism, Fig. 76, the angular velocity of the crank being known, it is required to find the angular velocity of the connecting rod. Again:

$$\frac{\omega_{31}}{\omega_{21}}=\frac{12-23}{13-23}.$$

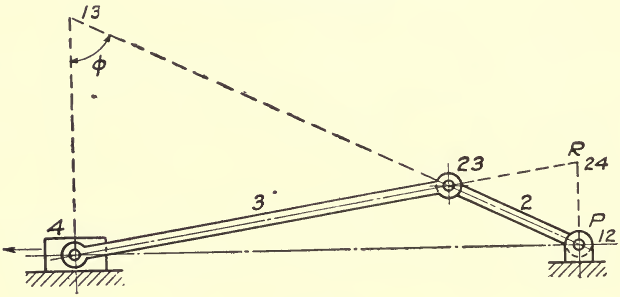


FIG. 76.

44. Special Cases.—(*a*) one of the centers involved may fall at infinity. For example, in the slider crank mechanism, Fig. 76, the center *14* lies at an infinite distance. Therefore:

$$\frac{\omega_{41}}{\omega_{21}}=\frac{12-24}{14-24}=\frac{PR}{\infty}=0.$$

In other words the cross-head, link *4*, has no angular velocity—a result which is self-evident.

(*b*) The instantaneous center of relative motion may fall at infinity. In the Whitworth quick-return motion shown in skeleton form in Fig. 77, there is a sliding pair between links *3* and *4*.

$$\frac{\omega_{31}}{\omega_{41}}=\frac{14-34}{13-34}=\frac{\infty}{\infty}.$$

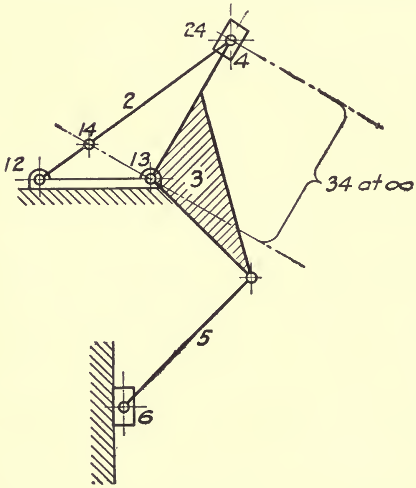


FIG. 77.

The angular velocity ratio is therefore apparently indeterminate. The distance *14–34* may be written (*14–13*)+(*13–34*).

Hence

$$\frac{\omega_{31}}{\omega_{41}}=\frac{14-13}{13-34}+\frac{13-34}{13-34}=0+1.$$

That is, *3* and *4* have the same angular velocity—again a self-evident conclusion.

(*c*) The ratio $\frac{rs-st}{rt-st}$ may be indeterminate. In the reverted epicyclic gear train, Fig. 68:

$$\frac{\omega_{41}}{\omega_{21}}=\frac{12-24}{14-24}=\frac{0}{0}.$$

To evaluate this indeterminate result write:

$$\frac{\omega_{41}}{\omega_{21}}=\frac{\omega_{31}}{\omega_{21}}\times\frac{\omega_{41}}{\omega_{31}}=\frac{12-23}{13-23}\times\frac{13-34}{14-34};$$

$$=\frac{r_a+r_b}{r_b}\times\frac{r_a-r_e}{r_e}=\left(1+\frac{r_a}{r_b}\right)\left(\frac{r_a}{r_e}-1\right);$$

or since the radii are proportional to the numbers of teeth,

$$\frac{\omega_{41}}{\omega_{21}}=\frac{\omega_{ea}}{\omega_{ba}}=\left(1+\frac{a}{b}\right)\left(\frac{a}{e}-1\right)=\left(\frac{a}{e}+\frac{a^2}{be}-\frac{a}{b}-1\right)=\left(\frac{ab+a^2-ae}{be}-1\right)$$

$$=\frac{a(a+b)-ae}{be}-1=\frac{a(d+e)-ae}{be}-1=-1+\frac{ad}{be},$$

which agrees with the value found by the analytical methods. It should be noted that this method does not give the direction of rotation.

45. Relative Angular Velocity.—If it is desired to find the relative angular velocity between two moving links *S* and *T* the same line of reasoning may be employed.

$$\frac{\omega_{st}}{\omega_{rs}}=\frac{rs-rt}{st-rt}.$$

For example, consider two gears, links *2* and *3*, in mesh, Fig. 78.

$$\frac{\omega_{23}}{\omega_{21}}=\frac{12-13}{23-13}=\frac{r_2+r_3}{r_3}=\frac{r_2}{r_3}+1.$$

$$\omega_{23}=\frac{r_2}{r_3}\omega_{21}+\omega_{21}.$$

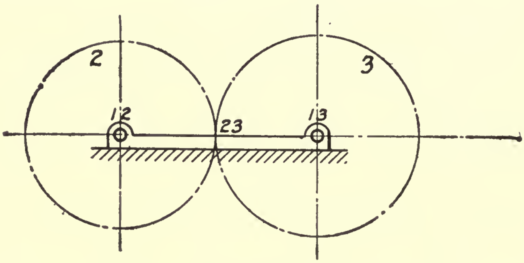


Fig. 78.

But

$$\frac{r_2}{r_3}\omega_{21}=-\omega_{31}.$$

Therefore

$$\omega_{23}=\omega_{21}-\omega_{31}.$$

For example, if

$$\omega_{21}=+4 \text{ radians per second,}$$

and

$$\omega_{31}=-5 \text{ radians per second,}$$

then

$$\omega_{23}=+4-(-5)=+9 \text{ radians per second.}$$

This equation may be generalized so as to cover any case, as follows:

$$\omega_{st}=\omega_{sr}-\omega_{tr}.$$

46. Linear Velocity Ratios.—In many cases it is required to find the ratio of the linear velocities of points on different links. Let A and B, Fig. 79, be points on links S and T respectively.

$$V_a=\text{velocity of } A=P_sA\,\omega_s$$

$$V_b=\text{velocity of } B=P_tB\,\omega_t$$

$$\frac{V_a}{V_b}=\frac{P_sA}{P_tB}\cdot\frac{\omega_s}{\omega_t}=\frac{P_sA}{P_tB}\cdot\frac{P_t-st}{P_s-st}.$$

Or

$$V_b = V_a \frac{P_tB}{P_sA} \cdot \frac{P_s - st}{P_t - st}.$$

This equation is perfectly general and may be applied to any plane mechanism.

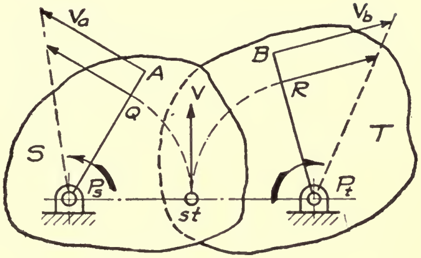


FIG. 79.

47. Special Cases.—(*a*) The two points considered may lie on the same link. In this case the equation reduces to

$$V_b = V_a \frac{PB}{PA},$$

where P is the instantaneous center.

(*b*) One of the centers P_s or P_t may lie at infinity. In this case if P_s is at infinity:

$$V_b = V_a \frac{P_tB}{P_t - st}.$$

For example, in the slider crank mechanism, Fig. 80, the center *14* is at infinity. Suppose the velocity of point A, say on the flywheel, to be known. To find the velocity of a point D on the piston write:

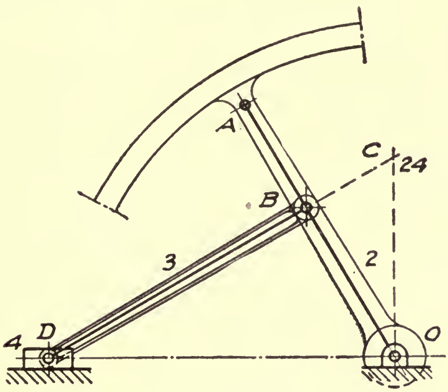


FIG. 80.

$$V_a = V_d \frac{OA}{OC}$$

since C is the center *24*.

We may interpret this result by noting that the center C (24) is a point which has the same velocity whether considered on link *2* or link *4*. Since every point on link *4* has the same velocity, the center C might be briefly described as a point on link *2*

which has the same velocity as link *4*. Considering C as a point on link *2*:

$$\frac{V_c}{V_a}=\frac{OC}{OA}.$$

Therefore,

$$V_c=V_d=V_a\frac{OC}{OA}.$$

(*c*) The center between the two moving links may lie at infinity. In this case the general equation reduces to:

$$V_b=V_a\frac{P_tB}{P_sA}.$$

For example, in the four-link mechansim, Fig. 81, the center *24* lies at infinity. Given the velocity of A on link *2* to find the velocity of B on link *4*. The center *14* is located at O.

$$V_b=V_a\frac{OB}{P_2A}.$$

This result may be interpreted as follows: Since links *2* and *4* have the same angular velocity (see Art. 44) the velocities of points on these links are directly

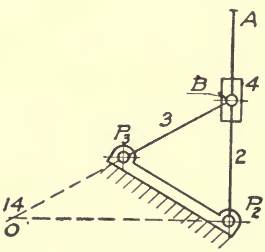


Fig. 81.

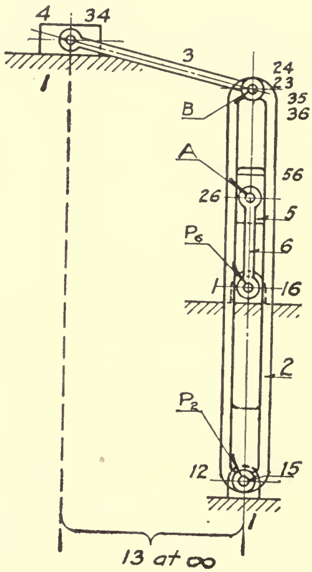


Fig. 82.

proportional to the distances of the points from their respective instantaneous centers.

(*d*) Sometimes the center of relative motion cannot be found by ordinary methods. In such cases special devices must be used.

For example, in the shaper mechanism shown in Fig. 42 there are two positions where the center *46* cannot be determined by the law of three centers. These positions are those where the center line of the oscillating lever is vertical, Fig. 82. In this case the centers are located as follows:

$$13 \left| \begin{matrix} 12\text{–}23 \\ 14\text{–}43 \end{matrix} \right\} \text{at infinity} \qquad 15 \left| \begin{matrix} 12\text{–}25 \\ 16\text{–}65 \end{matrix} \right\} \text{at } P_2 \qquad 24 \left| \begin{matrix} 12\text{–}14 \\ 23\text{–}34 \end{matrix} \right\} \text{at } B$$

$$26 \left| \begin{matrix} 12\text{–}16 \\ 25\text{–}56 \end{matrix} \right\} \text{at } A \qquad 35 \left| \begin{matrix} 13\text{–}15 \\ 23\text{–}25 \end{matrix} \right\} \text{at } B$$

$$36 \left| \begin{matrix} 13\text{–}16 \\ 23\text{–}26 \\ 35\text{–}56 \\ 34\text{–}46 \end{matrix} \right\}$$

Of these loci, the first three coincide with the vertical center line, and therefore have no intersection. The fourth contains the unknown center *46*.

$$46 \left| \begin{matrix} 14\text{–}16 \\ 24\text{–}26 \\ 54\text{–}56 \\ 34\text{–}36 \end{matrix} \right\}$$

Of these loci again three coincide with the vertical center line and the fourth is unknown. Therefore, the centers *36* and *46* are indeterminate.

The velocity of C can be determined from the following considerations: Since the center *13* falls at infinity all points on link *3* have the same velocity. Hence $V_c = V_b$. Considering B now as a point on link *2* and A as a point on link *5*.

$$V_b = V_a \frac{P_2B}{P_2A},$$

according to the result obtained in special case (*c*).

If desired, the location of *36* and *46* may now be found by a simple calculation.

48. Method of Relative Velocities.—While the method of instantaneous centers is a general one and may be applied to all plane mechanisms, many cases arise where its application is laborious and inconvenient. In mechanisms having a large number of links, the number of centers increases very rapidly, and many of them may fall at considerable distances. On these accounts the constructions become tedious, and the chances of inaccuracies become serious. For complex linkages, a more

direct and less troublesome method of determining velocities may be employed.

The *relative velocity* of a point A with respect to a second point B is defined as the vector difference between the velocities of B and A. In other words, it is that velocity, which added vectorially to that of B will give the velocity of A. In Fig. 83, V_a is the velocity of A, V_b that of B, and V_{ab} the relative velocity of A with respect to B.

The relative velocity is unaffected by any motion which is given simultaneously to both points. Thus the relative motion of the piston and cylinder, or the wrist pin and crank pin of a

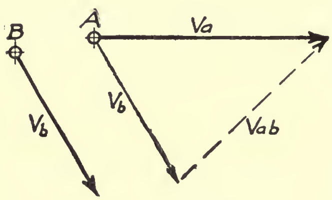


Fig. 83.

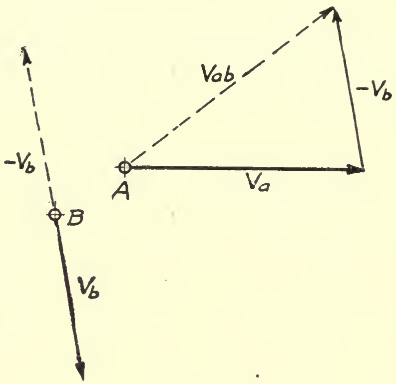


Fig. 84.

locomotive is independent of the travel of the locomotive along the track. In particular, the entire system may be given a velocity equal and opposite to that of B, as illustrated in Fig. 84. B is simultaneously given the velocities V_b and $-V_b$, and consequently remains at rest. A is given the velocities V_a and $-V_b$. The resultant of these is the relative velocity V_{ab}. This is readily seen to be the same result as that obtained in the previous figure. Therefore a second definition of relative velocity may be given as follows:

The relative velocity of a point A with respect to a second point B is the velocity which A would have if B were brought to rest, by imparting to both points a velocity equal and opposite to that of B.

49. Special Cases.—Two special cases of importance in kinematics require separate treatment.

(*a*) In the case where one link drives another by direct contact, the relative velocity of the two points which are in contact is along the common tangent. For if the relative velocity V_{ba} had any component along the common normal the bodies would cut into one another, or would separate. See Fig. 85.

(*b*) If the points A and B are connected by a rigid link the only motion A can have if B is brought to rest is a rotation about B as a center. Hence the relative velocity of A with respect to B must be in a direction at right angles to the line AB. See Fig. 86.

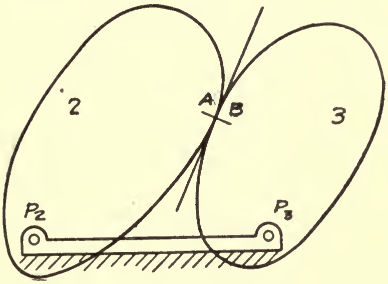


Fig. 85.

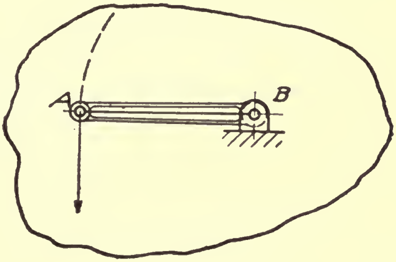


Fig. 86.

These principles are very important in the study of velocities in mechanisms.

50. Velocity Images.—Consider any link, Fig. 87, rotating about an instantaneous center P with angular velocity ω. The velocities of any points A, B, C, etc., of this link are proportional to the instantaneous radii, PA, PB, PC, etc., and are perpendicular to these radii.

From a common pole O lay off vectors Oa, Ob, Oc representing V_a, V_b, V_c, etc., in direction and magnitude. Then

$$Oa=\omega\times PA, \quad Ob=\omega\times PB, \text{ etc.}$$

Since
$$\angle aob = \angle APB$$

$$\Delta APB \text{ is similar to } \Delta Oab,$$

and therefore

$$ab=\omega AB.$$

Similarly

$$ac = \omega AC,$$

and

$$bc = \omega BC.$$

Hence the triangles ABC and abc are similar, whence follows the important theorem:

"If from a common pole vectors are drawn representing the velocities of the points of a rigid link, the ends of these vectors determine a figure similar to the link."

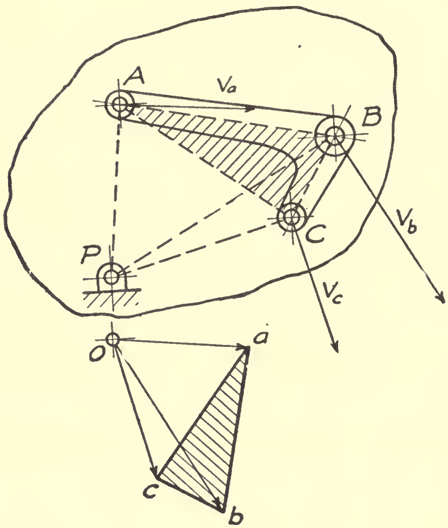


Fig. 87.

This figure is called the *velocity image* of the link. Evidently, the sides ab, bc, ac, are perpendicular to AB, BC, AC. This is plain from the geometry of the figure, or it may be proved independently—ab represents the velocity of A relative to B, since it is the vector which must be added to V_b in order to obtain V_a. This relative velocity must be perpendicular to AB. Similarly, bc is the velocity of C relative to A, and ca the velocity of A relative to C. It has been shown that $ab = \omega AB$. Hence follows another important theorem:

"The relative velocity between two points connected by a rigid link is perpendicular to the line joining the points, and is equal to the distance between the points multiplied by the angular velocity of the link. In case a link has a motion of translation its angular velocity is zero. In this case there is no relative velocity between the points of the link, and the velocity image reduces to a point.

51. Revolved Velocities.—In most cases it will be found more convenient to revolve the vectors representing the velocities through a right angle. If this is done, the sides of the image become parallel to the corresponding sides of the link. See Fig. 88.

52. Velocity Polygons.—If vectors are drawn from a common pole representing the velocities of all the points of a mechanism, then the resulting figure contains a velocity image of each link. The complete figure is called the *velocity polygon* for the mechanism. As stated in the preceding paragraph, it is usually more convenient to revolve all the vectors through 90°. Then the sides of each image are parallel to the corresponding sides of the links. Since the size of the image is proportional to the angular velocity of the link, if follows that the larger link may be represented in the polygon by the smaller image.

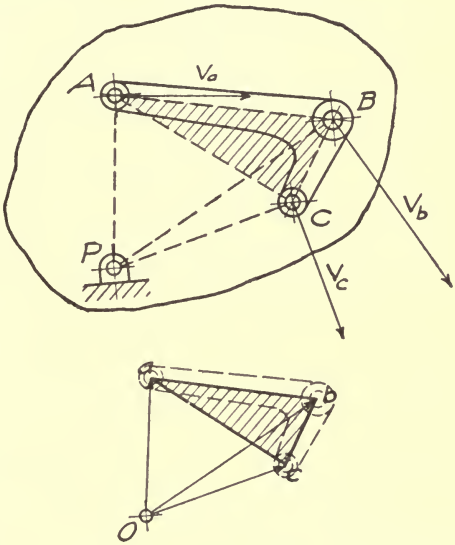


Fig. 88.

For finding the velocities of different points the method of relative velocities is employed. For this purpose an equation of the form

$$V_y = V_x + V_{yx},$$

is used. The plus sign in this equation represents vectorial addition.

53. Examples.—(*a*) As a simple example, consider the pair of cams, Fig. 89. Given the velocity of A on link *2* represented by the vector V_a. It is required to find the velocity of B on link *3*. From a pole O draw a vector Oa to represent the velocity of A

revolved through 90°. The velocity of C is made up of the velocity of A and the velocity of C relative to A, thus

$$V_c = V_a + V_{ca}.$$

The relative velocity is perpendicular to AC, or when revolved through 90° is parallel to AC. Through a draw a line parallel to AC. The end of the vector representing the velocity of C lies on this line. The direction of V_c is perpendicular to P_2C. Therefore draw a line through O parallel to P_2C. The intersection c of this line with the line drawn through a determines Oc as the velocity of C.

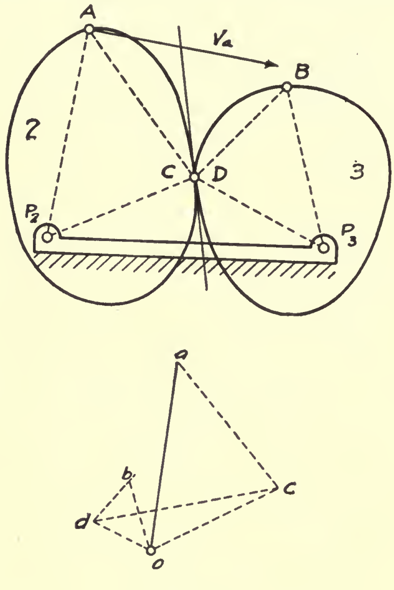


Fig. 89.

The velocities V_d and V_c are connected by the relation:

$$V_d = V_c + V_{dc}.$$

The direction of V_d is perpendicular to P_3D, and the direction of V_{dc} is along the common tangent. Therefore, if a line be drawn through O parallel to P_3D, and a line through c parallel to the common normal, the intersection of these lines determines the velocity of D. Again

$$V_b = V_d + V_{bd}.$$

The directions of V_b and V_{bd} are known. Therefore, if lines be drawn through O parallel to P_3B, and through d parallel to DB, the intersection b of these lines determines the velocity of B as Ob. Note that all the velocities in the polar diagram are drawn perpendicular to their real directions. To find the relative velocities of any two points as A and B, simply draw a vector connecting the corresponding points a and b of the polar diagram. The

triangles Oac and Odb are images of P_2AC and P_3DB respectively. The image of the stationary link is simply the pole O.

(*b*) As a second example take the pair of gears *2* and *3*, Fig. 90. From a pole O draw a vector Oa parallel to P_2A to represent the velocity of A. The image of gear *2* is a circle about O as center with radius Oa. The velocity of C is found by drawing the diameter aOc. As C has the same velocity as D, the image of D coincides with that of C. The

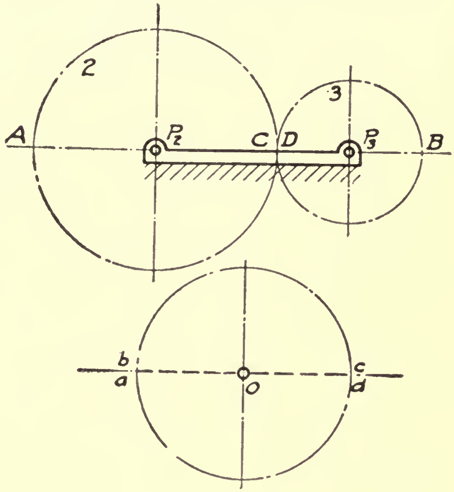


FIG. 90.

image of gear *3* is a circle about O having a radius Oc—that is, the images of gears *2* and *3* coincide. The image of B coincides with that of A. It will be noted that the image of gear *2* is erect while that of gear *3* is inverted. The significance of this fact is simply that the gears rotate in opposite directions.

(*c*) Another example is shown in the epicyclic gear train, Fig. 91. If the angular velocity of the arm *2* is known, the velocity of A is readily found. From a pole O draw a vector Oa equal to V_a. This vector is the image of P_2A, that is link *2*. The image of link *3* is a pair of circles having a common center at a. As point B has no velocity, its image lies at O. This determines the radius of the circle representing the smaller of gears *3*. The radius of the image of the larger gear is found by proportion and thus c is located. The image of the gear *4* is a circle whose center is O and whose radius is Oc. The fact that this image is inverted shows that *4* rotates in the direction opposite to that of *2*. It should be noted that

$$Oa = \omega_2 \cdot P_2A = \omega_3 \cdot BA,$$

and

$$Oc = \omega_4 \cdot P_2C\cdot$$

The reader may profitably study this construction, and see whether he can prove that the values of the ratios $\frac{\omega_4}{\omega_2}$ and $\frac{\omega_3}{\omega_2}$, found by this method, are the same as those found in Arts. 39 and 43.

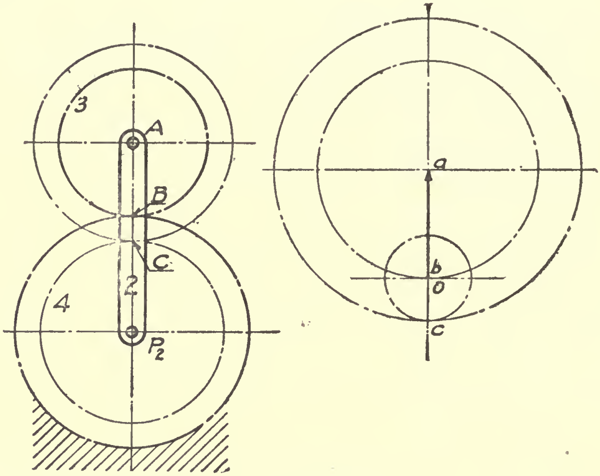


FIG. 91.

54. Velocity Polygons for Linkages.[1]—Consider the four-link chain, Fig. 92. Given the velocity of point A, it is required to find the velocities of points B, C, and D. From a pole O lay off a vector $Oa = V_a$.

$$V_b = V_a + V_{ba}.$$

V_b and V_{ba} are known in direction. Therefore draw through O a line parallel to P_4B, and through a a line parallel to AB. The intersection of these lines locates the image of B. To find the image of C simply locate on ab a point c so that

$$\frac{AC}{AB} = \frac{ac}{ab}.$$

[1] It is well at least in the first few polygons constructed to draw vectors from the different points of the machine representing the velocities of these points. This will give the student a clear idea of what is actually taking place in the machine.

To locate the image of D draw through b a line parallel to BD, and through a a line parallel to AD. Then Oa is the image of link *2*, Ob of link *4*, and $abcd$ of link *3*. The velocities of C and D are Oc and Od respectively.

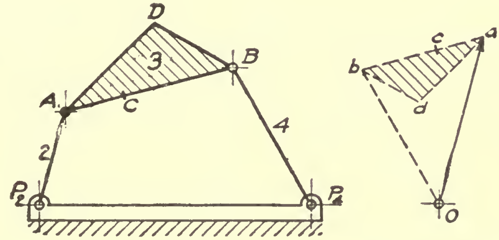


Fig. 92.

55. Applications to More Complex Linkages.—A great many mechanisms are built up from the simple arrangements shown in the preceding examples by merely adding extra gears or extra links two at a time. In such mechanisms all the velocities can be found by applying the preceding principles.

(*a*) **Joy Locomotive Valve Gear.**—In the steam engine mechanism, Fig. 93, let the velocity of the crank pin A be known. From

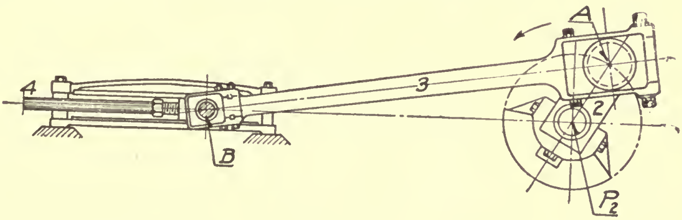


Fig. 93.

a pole O, Fig. 95, draw a vector Oa equal Va revolved through 90° as shown.

$$V_b = V_a + V_{ba}.$$

To determine V_b draw a vertical line through O and a line through a parallel to AB. Then Oa is the image of link *2*, ab that of link *3* and the point b that of link *4*.

Now suppose two additional links *5* and *6* to be added to the mechanism as shown in Fig. 94. The image of C is found by locating on ab a point c, such that $\frac{ac}{cb}=\frac{AC}{CB}$. To locate the image

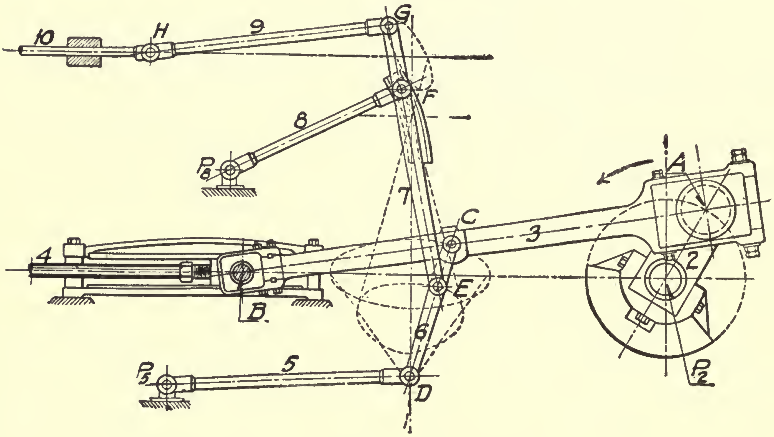


Fig. 94.

of D **draw** cd and Od parallel to CD and P_5D, as shown in Fig. 95. Now add two more links, *7* and *8* in Fig. 94. To locate the image of E, simply divide the line cd in the ratio $\frac{CE}{ED}=\frac{ce}{ed}$. To locate

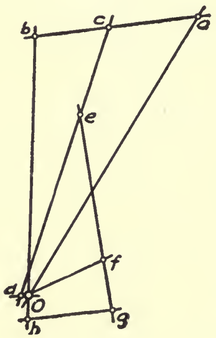


Fig. 95.

the image of F draw ef and Of parallel to EF and P_8F respectively. Finally, suppose two links *9* and *10* are added. To locate the image of G simply extend ef to g, so that $\frac{fg}{eg}=\frac{FG}{EG}$. Then locate the image of H by drawing gh parallel to GH, intersecting a vertical through O at h. Then Oh represents the velocity of the valve, link *10*. Each line of the velocity polygon, Fig. 95, is the image of the similarly lettered line of the mechanism.

(*b*) **Pilgrim Step Motion.**—In this mechanism, there is a four-link chain on which is superimposed a gear train, see Fig. 96.

The gear train consists of a small gear whose center is A, rigidly attached to the crank 2, and rotating with 2, an idle gear 5, whose

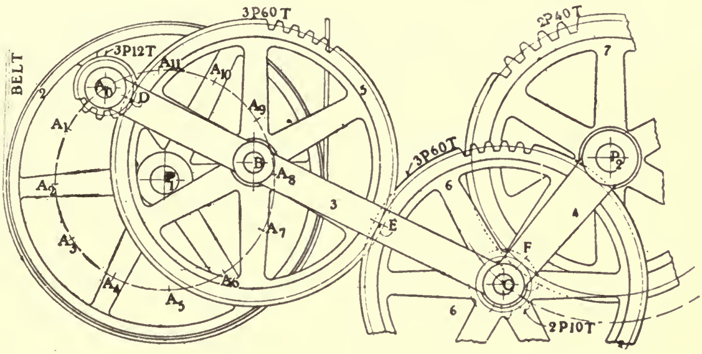


Fig. 96.

center is B on link 3, a compound gear 6, whose center is the moving pin C, and a driven gear 7, meshing with the smaller

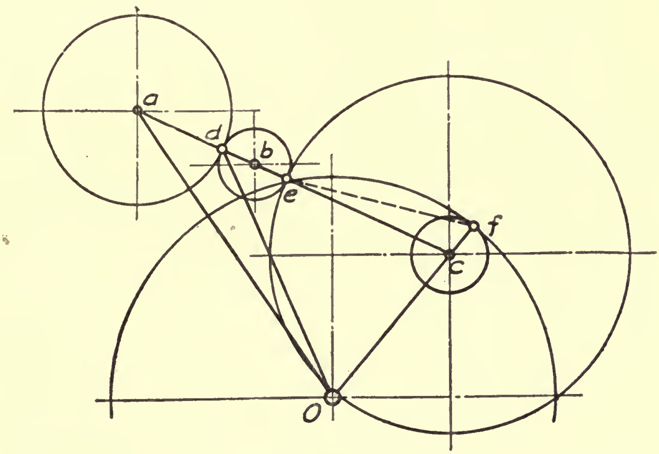


Fig. 97.

gear 6 and rotating about the fixed center P_4. To construct the velocity polygon, Fig. 97, assume the value of V_a to be represented by the vector Oa. Considering the four-link chain the velocity

of C is readily found. Then Oa is the image of P_1A and Oc the image of P_2C. To find the image of the different gears, first locate d by drawing Od parallel to P_1D. The image of the small gear *2* is a circle, having point a as a center and with a radius ad. Next, locate B, the center of gear *5*, by dividing the line ab so that $\frac{ab}{cb}=\frac{AB}{CB}$. The image of gear *5* is a circle about b as a center and with radius bd. The image of the larger gear *6* is found by drawing a circle of radius ce about c as a center. The image of the smaller gear may be found by proportion or by drawing the line ef parallel to EF. The image of gear *7* is a circle about O with radius Of.

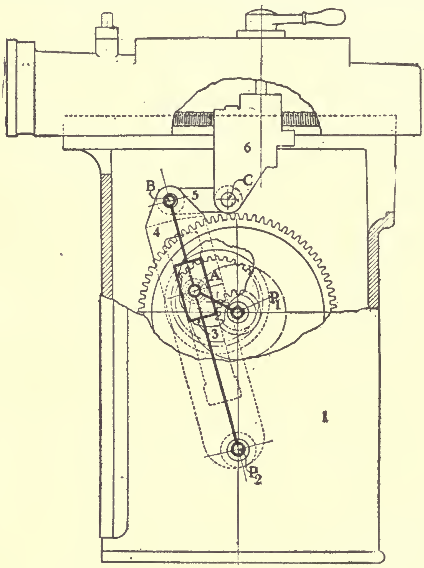


Fig. 98.

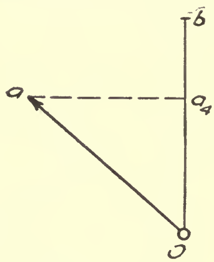


Fig. 99.

(*c*) **Shaper Mechanism.**—In the shaper mechanism, Fig. 98, the velocity of A, the center of the crank pin is known. Since the sliding block, link *3*, is equivalent to a binary link of infinite length it is impossible to find directly the velocity of a second point on link *3*, and the method of the preceding example therefore fails. It is possible, however, to determine the velocity of a point on link *4* which is located directly under A. Let the latter point be known as A_4. From the definition of relative velocity

$$V_{a_4}=V_a+V_{a_4a}.$$

From the pole O, Fig. 99, draw Oa equal V_a revolved through

90°. The direction of V_{a_4a} is along the center line of link 4, that is, parallel to P_2A. Through a draw a line perpendicular to P_2A. This line is a locus of the image of A_4. Through O draw a line parallel to P_2A. This line is also a locus for the image of A_4. The velocity of A_4 being thus determined the velocity of B is found by making $\frac{Ob}{Oa}=\frac{P_2B}{P_2A}$.

56. Special Constructions.—The methods developed in the preceding paragraphs can be employed in determining the velocities of all points in all mechanisms, in which the paths of the various points can be found by means of the straightedge and compass alone. Where this is not the case more complex methods must be employed.

The special method most frequently employed is known as the *Three-line construction.* Suppose that in a revolved velocity

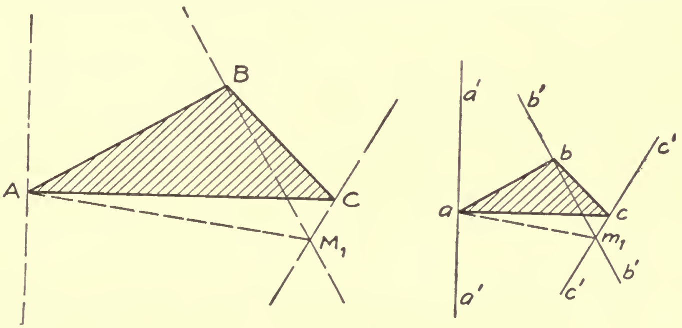


FIG. 100.

polygon the images of three points of a link ABC, Fig. 100, must lie on the three lines $a'a'$, $b'b'$, and $c'c'$ respectively. Let m_1 be the intersection of two of these lines, say $b'b'$ and $c'c'$. From B and C draw lines parallel to $b'b'$ and $c'c'$ respectively and let these lines intersect at M_1. From M_1 draw M_1A, and through m_1 draw a line m_1a parallel to M_1A. The intersection of m_1a with $a'a'$ locates the velocity image of A at a. From a draw ab and ac parallel to AB and AC respectively. Then b and c are the images of B and C. If the work is correctly done, bc will be parallel to BC.

Proof.—Imagine the link ABC expanded so as to include M_1. Then write

$$V_m=V_c+V_{mc}=V_b+V_{mb}.$$

V_{mc} (revolved) has the direction M_1C, that is, the direction $c'c'$. Hence, as the image of C lies on $c'c'$ the image of M_1 also lies on $c'c'$. Similarly, the image of M_1 must lie on $b'b'$. Therefore m_1 is the image of M_1. This determines the velocity of one point M_1 on the link. To find the velocity of a second point A note that

$$V_a = V_m + V_{am}.$$

V_m is known and V_{am} (revolved) has the direction M_1A. Therefore drawing m_1a parallel to M_1A the image of A is located at a. The images of B and C are located as usual by drawing ab and ac parallel to AB and AC.[1] It should be noted that either of the intersections m_2 or m_3 can be used as well as m_1. Corresponding points M_2 and M_3 on the link would then be located. The choice of the intersection is simply a matter of convenience. An example will make the use of this construction more clear.

47. Example. Stephenson Link Motion.—The Stephenson link mechanism, which is shown fully drawn in Fig. 37, is redrawn in its skeleton form in Fig. 101. In this mechanism there is a four-link chain, *2345*, but the stationary or reference link is not one of its sides.

Fig. 102 shows the velocity polygon which is arrived at as follows: If the velocity of the eccentric center A is known, the velocity of point B is readily found. It is required to find the velocities of points C, D and E.

Note that

$$V_c = V_a + V_{ca},$$

where V_{ca} revolved has the direction AC. This gives one line ac on which the image of C must be located; also since,

$$V_e = V_b + V_{eb},$$

where V_{eb} revolved has the direction BE, the image of E must lie on the line be. The direction of V_d must be parallel to P_6D. This gives a locus Od for the image of D.

[1] The validity of this construction can also be proved by showing that the triangle abc has its sides parallel to those of ABC. abc is therefore the image of ABC. As ab and ac are parallel to AB and AC by construction, it is necessary only to prove bc parallel to BC.

Choose the intersection of two of the lines *ac*, *be* and *Od*, say m_2. Prolong AC till it intersects P_6D at M_2, Fig. 101. Now imagine link *5* expanded so as to include the point M_2. Then,

$$V_{m_2}=V_d+V_{m_2d}.$$

It is readily seen that V_d and V_{m_2d} are in the same direction,

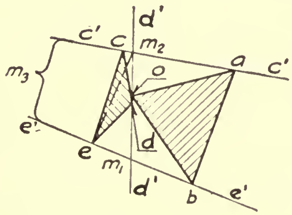


FIG. 102.

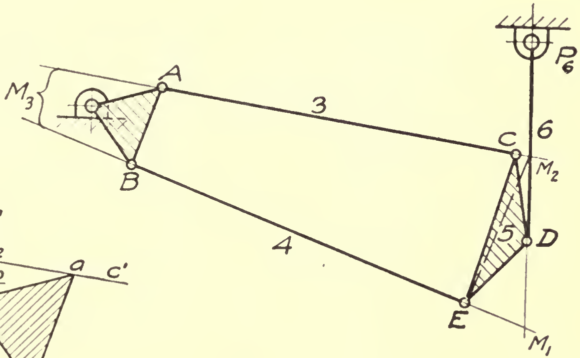


FIG. 101.

hence their resultant is also in this direction, and the locus *Od* is therefore a locus for the image of M_2.

Now

$$V_{m_2}=V_a+V_{m_2a}=V_c+V_{m_2c},$$

and

$$V_c=V_a+V_{ca}.$$

Therefore

$$V_{m_2a}=V_{ca}+V_{m_2c}.$$

Both V_{ca} and V_{m_2c} are in the same direction, and therefore *ac* is also a locus for the image of M_2. The intersection of *ac* and *Od* is consequently the image of M_2, and thus we have found the velocity of one point of link *5*.

To get the velocity of a second point E, it is only necessary to draw m_2e parallel to M_2E. The intersection of this line with *be* determines the velocity image of E. C and D are now readily located by drawing *ec* and *ed* parallel to EC and ED respectively.

The accuracy of the construction may be tested by noting that *cd* must be parallel to *CD*.

EXERCISES

1. Construct the velocity polygon for the Stephenson link motion, making use of point M_1 or M_3 instead of point M_2.

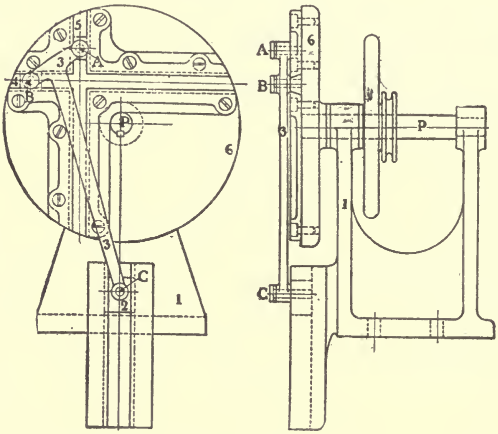


FIG. 103.

2. Construct the velocity polygon for the Wanzer needle bar mechanism, Fig. 103. The speed of rotation of the crank, link *6*, is known.

3. Construct the velocity polygon for the Watt engine, Fig. 104. The velocity of the crank pin *A* is known.

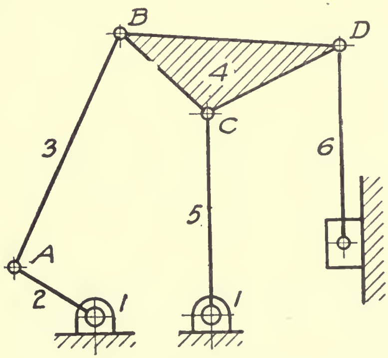


FIG. 104.

In some cases a mechanism is built on a four-link chain and the extra links are added in groups larger than two. In such cases the three-line construction is often useful.

4. Construct the velocity polygon for the skeleton mechanism, Fig. 105. Link *1* is fixed and the velocity of point *A* is known.

58. Combined Method of Instantaneous Centers and Relative Velocities.—In some cases a velocity polygon can be readily con-

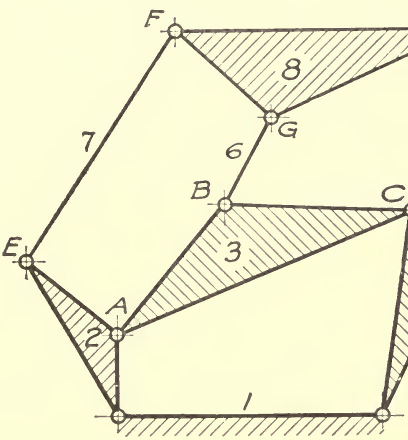


FIG. 105.

structed by the determination of certain centers, when it would be difficult or impossible to construct it by the method of relative velocities alone. As an example, consider the Wanzer needle-bar mechanism, Fig. 106. Suppose the velocity of

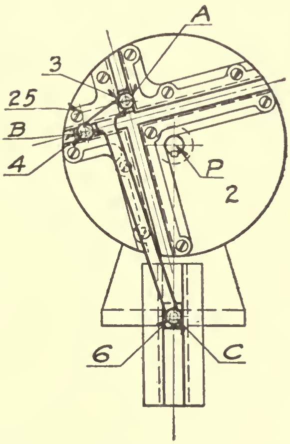


FIG. 106.

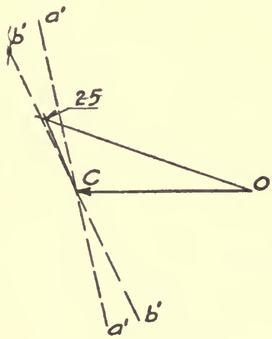


FIG. 107.

C is known and that it is required to construct the polygon. Let Oc, Fig. 107, represent the revolved velocity of C. $b'cb'$ is then a locus for the image of B, and $a'ca'$ is a locus for the image of A. Also

the direction of motion of any point on link *2* is known. With such data it is impossible to draw the polygon by any of the methods given previously. The construction is, however, easily completed by locating the center *25*, and remembering that the velocity of *25* is the same whether it be regarded as on link *2* or link *5*. Considering *25* as a point on link *2* its revolved velocity is parallel to $P_2 25$. A line through O in this direction is, therefore, a locus of *25*, that is, the image of *25*. Considering *25* as a point on link *5*, write

$$V_{25} = V_c + V_{25c}.$$

The revolved velocity V_{25c} is parallel to $C25$. This gives a second locus for $_{25}$ as shown. Having thus determined the velocity of one point on link *2*, the velocity of any other point on the same link is readily found. The completion of the polygon is left to the reader.

EXERCISE

5. In the Stephenson link mechanism, Fig. 101, the velocity of point D is known. Construct the velocity polygon.

59. Four-line Construction.—In some complex mechanisms a method involving both relative velocities and instantaneous centers can be employed. Suppose that one locus is known for the velocity image of each of four points, two of which are on a link S and two on a link T. If the instantaneous center ST is known the following construction will give the velocity images of the links: Let AB, Fig. 108, represent the link S and CD the link T and let Z be the instantaneous center ST. The velocity images of A, B, C and D are known to lie on the lines $a'a'$, $b'b'$, $c'c'$ and $d'd'$ respectively. Through A, B, C and D draw lines parallel to $a'a'$, $b'b'$, $c'c'$ and $d'd'$ respectively, Fig. 108. The lines through A and B intersect at X and those through C and D at Y. The lines $a'a'$ and $b'b'$ intersect at x and $c'c'$ and $d'd'$ at y. Connect X and Y to Z. Through x and y draw lines parallel to XZ and YZ respectively. These lines intersect at z. Lines drawn from z parallel to ZA, ZB, ZC and ZD locate the images of A, B, C and D on the lines $a'a'$, $b'b'$, $c'c'$ and $d'd'$.

Proof.—Consider links S and T extended so as to include the points X and Y respectively, and so that each includes the instan-

taneous center Z. Then the velocity of X is made up of the velocity of A and the velocity of X relative to A. The velocity of X relative to A is drawn parallel to XA, since X and A are on the same link. The velocity image of A lies on the line $a'a'$, and to find that of X we must add a component along the same line. Hence the velocity image of X lies also on line $a'a'$. Similarly the velocity of X must lie on the line $b'b'$, and therefore x is the velocity image of point X on link S. Similarly, y is the velocity image of point Y on link T. Hence the velocity of one point on each link is found. The instantaneous center Z has the same velocity whether considered on link S or link T. Considered as a point on link S the velocity of Z is made up of the velocity of X, which is known, and a velocity in the direction XZ. Considered as a point on link T the velocity of Z is made up of the velocity of Y and a component in the direction YZ. Hence z is the velocity image of the instantaneous center Z. Now the velocities of two points X and Z of link S and Y and Z of link T are known. To find the complete images of the links it is necessary only to complete the figures $zabx$ and $zcdy$, with sides parallel to $ZABX$ and $ZCDY$ respectively.

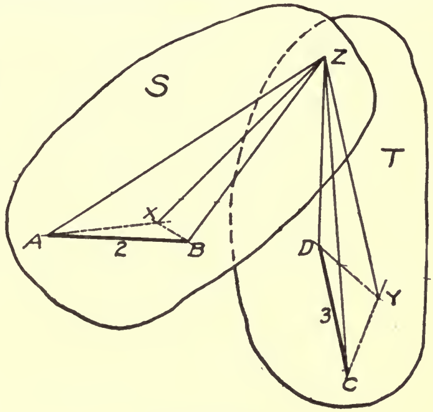


Fig. 108.

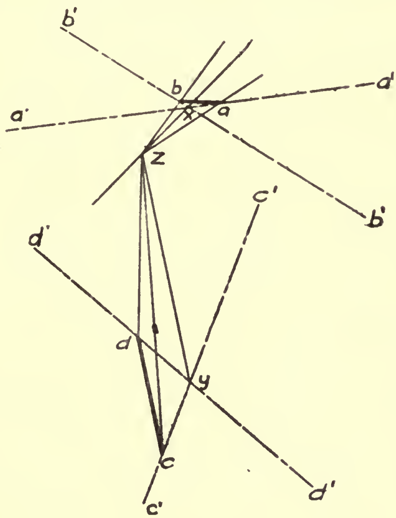


Fig. 109.

EXERCISE

6. Construct a velocity polygon for the Walschaert valve gear, Fig. 110. The velocity of the crank pin is known.

60. Velocity Curves.—In many cases it is desirable to know the velocities of different parts of a machine, not only for one position, but for every possible position. Usually, some one link can be assumed to run at constant speed. For example, in the slotter mechanism, Fig. 111, the crank, link *2*, may be assumed to run uniformly. It is desired to determine the variation in the velocity of the slide, link *6*, during one revolution of the crank.

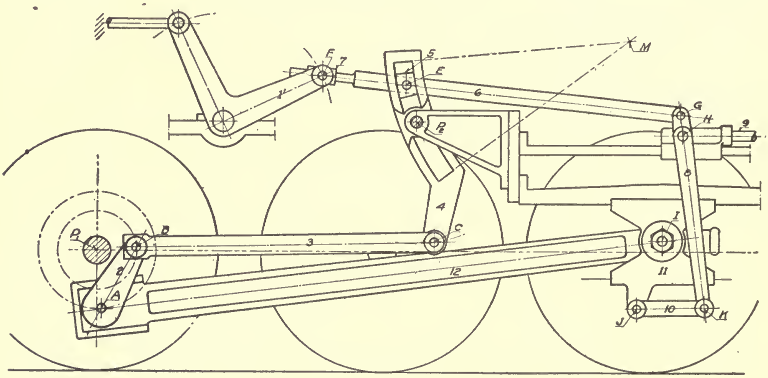


Fig. 110.

The mechanism can be drawn in skeleton form for each 15° through which the crank revolves. For each such configuration the velocity of the ram can be found either by instantaneous centers or by velocity polygons. Then at each position of the ram a line perpendicular to the stroke can be drawn representing the instantaneous velocity for that position. Velocities upward may be drawn toward the right, and those downward toward the left. In this way a curve may be laid down, showing the velocity for each position of the ram. Such a curve is shown in Fig. 111.

A second curve can be drawn to show the variation of the velocities during one revolution, by drawing a base line and dividing it into twenty-four equal parts, each space representing 15° of revolution of the crank. As the crank is supposed to rotate

at constant speed these spaces represent equal intervals of time. Such a curve is shown in Fig. 112. Some interesting conclusions may be drawn from this diagram. As the ordinate is velocity and the abscissa time, the slope of the curve represents the acceleration of the moving part. Also the area enclosed between the curve and the base line gives the distance traveled in any interval of time. The area above the base line represents the length of the cutting stroke, and that below the base line the length of the return stroke. Evidently these areas should be equal. If the tool cuts uniformly at maximum speed the amount of cutting, in the interval of time required for one revolution would be represented by the area of the rectangle *EFKM*. The actual cutting is represented by the figure *EGHDE*. The ratio $\frac{EGHDE}{EFKM}$ may be regarded as a sort of "Time Efficiency" of the machine. This furnishes a useful criterion for comparing the effectiveness of various types of quick-return motions.

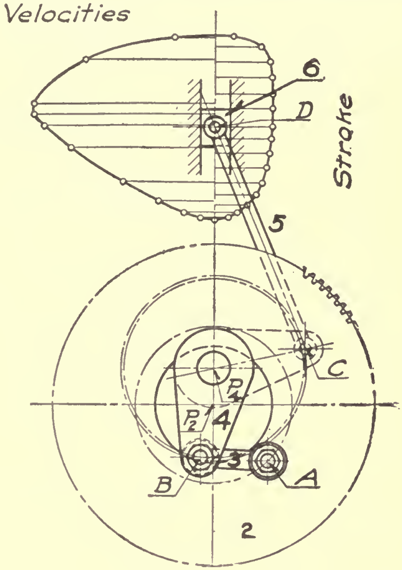


Fig. 111.

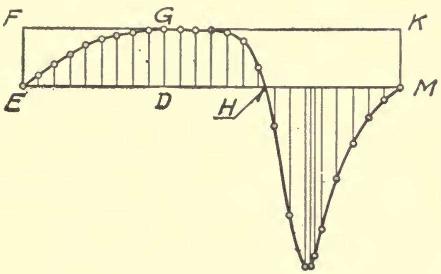


Fig. 112.

EXERCISE

7. Construct velocity curves and determine time efficiency of the quick-return motion shown in Fig. 98. The velocity of gear 2 is constant.

CHAPTER IV

ACCELERATION IN MECHANISMS

63. Introductory.—The study of accelerations in mechanisms will be carried out by methods somewhat analogous to those used in finding velocities. Analytical methods will be used in some simple cases, but for complex linkages graphical constructions are employed. The constructions for accelerations are more difficult than those for velocities and must be studied with great care.

64. General Principles.—Acceleration is defined as the rate of change of velocity. The general method of finding the acceleration of a point is as follows: Let V_1, Fig. 113, denote the velocity of a point at any instant, and V_2 the velocity after an interval of time Δt. Then $V_2 \rightarrow V_1$ denotes the vector difference between V_2 and V_1, that is, the velocity which must be combined with V_1 in order to give V_2 as a resultant. The quotient $\frac{V_2 \rightarrow V_1}{\Delta t}$ is the mean rate of change of velocity, or in other words, the *mean acceleration,* and the limit of this quotient as Δt approaches zero is the *instantaneous acceleration.* Denoting the acceleration by A we have

$$A = \underset{(\Delta t = 0)}{\text{limit}} \frac{V_2 \rightarrow V_1}{\Delta t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If, instead of the vector difference, we take the scalar difference of velocities we get quite a different quantity, the *rate of change of speed.* Denoting this by f

$$f = \underset{(\Delta t = 0)}{\text{limit}} \frac{V_2 - V_1}{\Delta t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In Fig. 114 let OA represent the initial velocity V_1 and OB the velocity V_2 that the point has attained after an interval of time Δt. Then the closing side AB is the velocity component that must be added to OA to give OB. Let OC be laid off equal

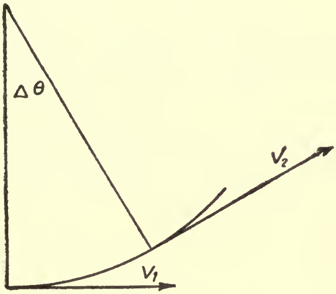


FIG. 113.

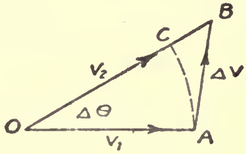


FIG. 114.

to OA: then $CB=OB-OC=V_2-V_1$. The limit of $\left(\frac{AB}{\Delta t}\right)$ gives the acceleration or rate of change of velocity, while the limit of $\left(\frac{CB}{\Delta t}\right)$ gives merely the rate of change of speed.

Now vector AB has the components AC and CB; that is, by vector addition $AB=AC+CB$.

Whence

$$\frac{AB}{\Delta t}=\frac{AC}{\Delta t}+\frac{CB}{\Delta t}$$

and

$$\text{lim.}\left[\frac{AB}{\Delta t}\right]=\text{lim.}\left[\frac{AC}{\Delta t}\right]+\text{lim.}\left[\frac{CB}{\Delta t}\right]. \quad . \quad . \quad . \qquad (3)$$

In the limit, the angle $COA=O$, and angle $ACB=90$, thus the components are at right angles.

$$\text{lim.}\left[\frac{CB}{\Delta t}\right]=\text{lim.}\left[\frac{V_2-V_1}{\Delta t}\right]=\frac{dV}{dt}=\frac{d^2s}{dt^2}=A_t,$$

$$\text{lim.}\left[\frac{AC}{\Delta t}\right]=\text{lim.}\left[\frac{2V_1\sin\frac{\Delta\theta}{2}}{\Delta t}\right]=V_1\,\text{lim.}\left[\frac{\Delta\theta}{\Delta t}\right]=V_1\omega,$$

in which θ is the angle between the directions of the velocities V_1 and V_2, and ω is the rate of change of direction of motion equal

$\frac{d\theta}{dt}$. If r is the radius of curvature of the path at the first instant, $V_1 = r\omega$ and

$$\lim.\left[\frac{AC}{\Delta t}\right] = V_1\omega = V_1\frac{V_1}{r} = \frac{V_1^2}{r}.$$

Since the discussion applies to every point in the path, the subscripts may be dropped. The component $\frac{V^2}{r}$ is in the limit perpendicular to the direction of the velocity of the moving point, in other words, normal to the path. Hence, it is called the *normal* or *radial acceleration* of the point. The other component $\frac{dV}{dt} = \frac{d^2S}{dt^2}$ has the direction of the tangent and is therefore called the *tangential acceleration*. Denoting these components by A^n and A^t respectively, the resultant or total acceleration A is given by the equation

$$A = \sqrt{A^{n^2} + A^{t^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the angle ϕ which A makes with the normal to the path is given by the relation[1]

$$\tan\phi = \frac{A^t}{A^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

65. Translation.—A plane system has a motion of translation when the paths of all points are equal and parallel. These paths may be either straight lines or curves. Evidently, since all points have precisely the same motion, they have the same velocity and also the same acceleration.

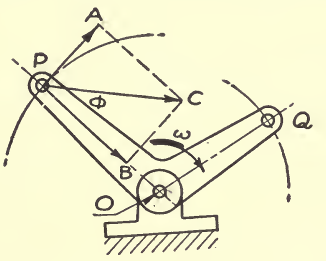


FIG. 115.

66. Rotation about a Fixed Axis.—Suppose a rigid system to rotate about a fixed axis, and let ω denote the angular velocity at a given instant. In Fig. 115 let O be the center of rotation and P and Q be points of the system. The point P describes a

[1] Unless otherwise stated, all additions and subtractions of velocities and accelerations will be considered as vectorial, and the usual symbol of vectorial addition and subtraction will be omitted.

circle with $OP=r$ as a radius. The linear velocity of P (not shown in Fig. 115) is

$$V=OP\times\omega=r\omega. \qquad (1)$$

The tangential acceleration represented by PA is

$$A^t=\frac{dV}{dt}=\frac{rd\omega}{dt}=r\omega', \qquad (2)$$

where ω' denotes the angular acceleration. The normal component represented by PB has the magnitude

$$A^n=\frac{V^2}{r}=\frac{(r\omega)^2}{r}=r\omega^2. \qquad (3)$$

For the angle ϕ we have

$$\tan\phi=\frac{PA}{PB}=\frac{r\omega'}{r\omega^2}=\frac{\omega'}{\omega^2}. \qquad (4)$$

Equation (4) shows that the angle ϕ is independent of r and is the same for every point.

The total acceleration has the magnitude

$$A=\sqrt{(r\omega')^2+(r\omega^2)^2}=r\sqrt{\omega'^2+\omega^4}. \qquad (5)$$

Equation (5) shows that the acceleration of a point is proportional to the distance r between the point and axis of rotation.

The existence of the normal component of the acceleration occasions the fundamental difference in the treatment of accelerations and velocities. As there is no component of the velocity normal to the path, we can easily determine the direction of the velocities of different points. This is not true of accelerations. It is this fact which makes the determination of accelerations more difficult than that of velocities. It should be noted that the tangential acceleration is due to change of speed, normal acceleration to change of direction.

67. Combined Motions. Coriolis' Law.—Frequently it is convenient to consider the motion of a system relative to a fixed body as composed of two motions: (1) The motion of a first system relative to a second moving system; (2) The motion of the second

system relative to the fixed body. As an example, take the motion of the shaft governor. Fig. 116. Consider first the motion of the governor relative to the flywheel as though the wheel were at rest, and then consider the motion of the wheel relative to the fixed bed. The combination of the two motions gives the absolute motion of the governor weight or the motion relative to the bed. Any point of the given system, S_1, may therefore be considered as moving in a curve mx in a second system S_2, while the curve at the same time moves in the fixed system S_3. The velocity of a point of system S_1 is the resultant of its velocity along the curve and the velocity of the coincident point of the curve. Thus, in Fig. 116, the point P moves along the curve m with the velocity u, while the point of the curve coincident with P has in the fixed system a velocity w. That is, if the curve were at rest, the point P would have the velocity u alone, while if P were fixed on the curve it would have the velocity w alone, due to the motion of the curve. It has been shown that the actual velocity of the point P relative to the fixed system is the resultant of u and w.

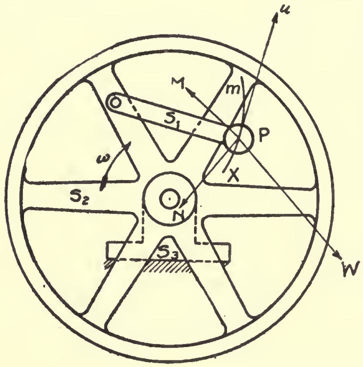


FIG. 116.

Consider the actual acceleration of the moving point. Suppose the vector PM to represent the acceleration of P due to its motion along the curve m, that is, the acceleration of P if the curve m were at rest, and suppose the vector PN to represent the acceleration of the point m coincident with P due to the motion of the curve in the fixed system. If the same law holds for accelerations as for velocities, the actual acceleration at P is the resultant of the relative acceleration PM and the acceleration PN of the coincident point. The problem now before us is to determine if such is the case. Fig. 117 shows the general case where the curve has any motion. In Fig. 117 m_1 and m_2 represent consecutive positions of the curve m, and P is the instantaneous center of the motion of m at the instant. A point at S' moves

in the curve with a velocity u_1 and the coincident point A_1 of the curve moves with a velocity Va_1 perpendicular to the instantaneous radius.

The point S moves along the curve with variable velocity so that if the curve remained stationary S would have a relative velocity u_2' after an interval of time Δt. But in this interval of time the curve has revolved around P through an angle $\Delta\theta$ and the point A_1 of the curve has reached A_2, and has attained a velocity Va_2. The point B_1 has reached B_2 and its velocity

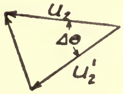


Fig. 118.

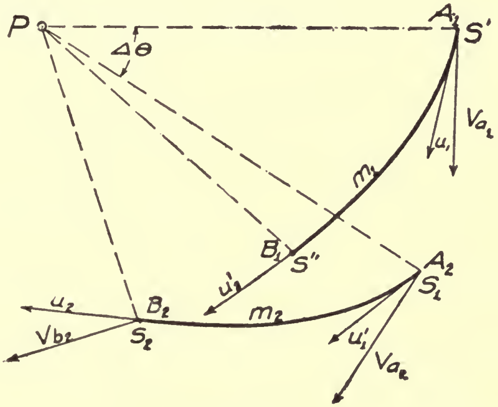


Fig. 117.

is V_{b_2}. If V_1 be the total velocity of the point S at the beginning of the interval of time Δt and V_2 its velocity at the end of that time. Then

$$V_1 = V_{a_1} + u_1$$

$$V_2 = V_{b_2} + u_2$$

$$V_2 - V_1 = V_{b_2} - V_{a_1} + u_2 - u_1$$

$$= V_{a_2} + V_{b_2a_2} - V_{a_1} + u_2 - u_1$$

Now $V_{a_2} - V_{a_1} = \Delta V_a =$ change of velocity of point A on curve m. Hence

$$V_2 - V_1 = \Delta V_a + V_{b_2a_2} + u_2 - u_1.$$

u_2' is numerically equal to u_2 but differs in direction by the angle $\Delta\phi$. (See Fig. 118.) Hence,

$$u_2 = u_2' + 2u_2 \sin \frac{\Delta\theta}{2}.$$

Hence,

$$V_2 - V_1 = \Delta V_a + V_{b_2a_2} + u_2' + 2u_2 \sin \frac{\Delta\theta}{2} - u_1.$$

Now $u_2' - u_1 = \Delta u =$ change of velocity considering the curve stationary.

Therefore

$$\Delta V = V_2 - V_1 = \Delta V_a + \Delta u + V_{b_2a_2} + 2u_2 \sin \frac{\Delta\theta}{2},$$

$$V_{b_2a_2} = A_2B_2\omega \text{ (see Art. 50).}$$

Therefore

$$\frac{\Delta V}{\Delta t} = \frac{\Delta V_a}{\Delta t} + \frac{\Delta u}{\Delta t} + \frac{V_{b_2a_2}}{\Delta t} + \frac{2u_2 \sin \frac{\Delta\theta}{2}.}{\Delta t}$$

In the limit,

$$A = \frac{dV_a}{dt} + \frac{du}{dt} + \frac{udt\omega}{dt} + \omega u_2,$$

since $A_2B_2 = udt$.

Now the component $V_{b_2a_2}$ is at right angles to A_2B_2 (that is, normal to the curve in the limit) and in the direction given by revolving the curve about A_2 in the sense ω. In the limit the component $u_2 - u_2'$ is in the same direction and sense, $\frac{dV_a}{dt}$ is by definition A_m, $\frac{du}{dt}$ is A_r.

Therefore

$$A = A_m + A_r + 2u\omega.$$

The law thus expressed mathematically is known as *Coriolis' Law.* The third component $2u\omega$ is called the *compound supple-*

mentary acceleration. The direction of this component, as we have seen, is perpendicular to that of the relative velocity u. The following rule will always give the proper sense of $2u\omega$. The sense of $2u\omega$ is such that, considered as a force, applied at the end of the vector representing u, it would rotate the vector, in the sense of ω, the angular velocity of the path m about its instantaneous center. Thus in Fig. 119 suppose the curve m to be rotating clockwise about P while the point S has at the same time the velocity $u = SA$ in the curve. A force applied at the end A of this vector must have the sense AB to turn the vector clockwise about A; the acceleration $2u\omega$ has the same sense as AB and passes through S perpendicular to SA.

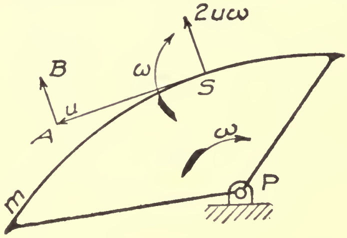


Fig. 119.

The compound supplementary acceleration is the result of two factors.

(1) The velocity u_2 is measured relative to a new point B on the curve which has a velocity different from that of A. Therefore the difference in velocity between B and A enters into the change of velocity of the point S.

(2) The direction of velocity u is along the tangent to the curve m. If this tangent turns through an angle $\Delta\theta$, the direction of u is changed by the same angle. This change of direction implies anothei component of the acceleration.

Special Case.—If the motion of the system carrying the curve m is a translation, then the points A and B have the same velocity. Also the angle θ through which the curve turns is *zero*, and therefore the compound supplementary acceleration disappears, as is also evident from the fact that $\omega = 0$. This is equally true whether the translation is rectilinear or curvilinear.

As an example consider the motion of a wheel or circular disk rolling on a plane surface. A point P of the disk actually de-

scribes the arc of a cycloid (Fig. 120), but we may consider that P moves in a circle m which at the same time has a motion of translation parallel to AB. Suppose the center O moves with a *constant velocity* w; then P moves with a constant speed u in the circle m and the circle has a translation with constant velocity. Under these conditions, the relative acceleration $A_r = \frac{u^2}{PO}$ and is directed toward the center O, and the acceleration of the point of the curve m coincident with P, that is A_m is *zero*. Hence, the actual acceleration of P is

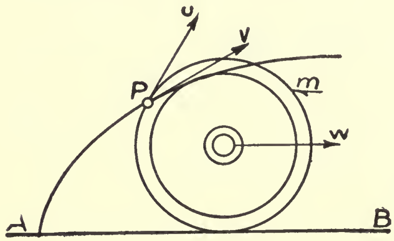


Fig. 120.

$$A_p = A_r + A_m = \frac{u^2}{OP} + O = \frac{u^2}{OP},$$

which is the same as if the disk were rotating with the same angular velocity about O as a fixed center.

As a second example, let a point move with constant speed in a helix, say the mean helix of a screw thread. This motion is equivalent to the motion of the point in a circle with a constant speed u, while the circle at the same time has a motion of translation perpendicular to its plane with a uniform velocity w. The relations between u, w, and V are as follows:

$$V = \sqrt{u^2 + w^2},\ u = V \cos\alpha;\ \text{and } w = u \tan\alpha,$$

where α is the helix angle, and V is the actual velocity of the point in the helix.

If r is the helix radius the relative acceleration is

$$A_r = \frac{u^2}{r} = \frac{V^2 \cos^2\alpha}{r}.$$

The acceleration A_m of the coincident point on the circle is zero, hence,

$$A_p = A_r = \frac{V^2 \cos^2\alpha}{r}.$$

68. Application of Coriolis' Law.—(*a*) The following numerical example will serve to illustrate the use of Coriolis' Law. In Fig. 121 the system S_2 rotates about the fixed point O and a second system S_1 rotates about a point C of the first system. The angular velocity of S_2 about O or ω is 5 radians per second, that of S_1 about C or Ω, is 8 radians per second. Both ω and Ω are constants. $OC = 13$ feet. Find the acceleration of the point P of S_1 knowing that $PC = 5$ feet and $PO = 12$ feet. The component A_r, the acceleration of P relative to system S_1 is

$$A_r = CP\Omega^2 = 5 \times 8^2 = 320 \text{ feet per second per second.}$$

This component has the direction PC.

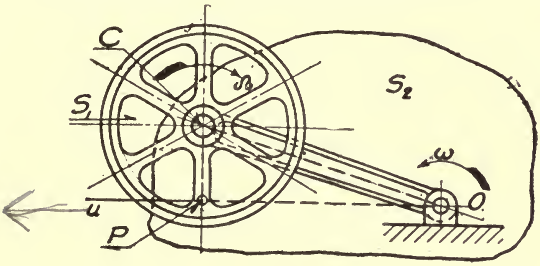


Fig. 121.

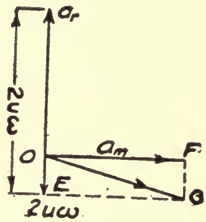


Fig. 122.

The component A_m, the acceleration of the point P of the system S_2, is

$$A_m = OP\omega^2 = 12 \times 5^2 = 300 \text{ feet per second per second.}$$

This component has the direction PO.

To get the component $2u\omega$, we have $u = CP\Omega = 5 \times 8 = 40$ feet per second, and $\omega = 5$; hence

$$2u\omega = 2 \times 40 \times 5 = 400 \text{ feet per second per second.}$$

This component is perpendicular to u in the direction CP.

The three components are shown in Fig. 122. Since A_r and $2u\omega$ are opposite but in the same line they may be added algebraically, giving as their sum 80 feet per second per second, represented by OE. OE combined with $A_m (= OF)$ gives a resultant represented by the vector OG.

$$A = \sqrt{80^2 + 300^2} = 310 \text{ feet per second per second.}$$

The angle between the direction of A and the radius OP, Fig. 121, is

$$\tan^{-1}\frac{80}{300}=15^\circ \text{ nearly.}$$

EXERCISES

1. In Fig. 123, gear *1* is stationary and gear *3* rolls around the circumference of gear *1*, the two gears being held in mesh by the arm link *2*.

Let gear *1* be 8 inches in diameter,

gear *3* be 4 inches in diameter,

and ω = angular velocity of link *2* = 10 radians per second.

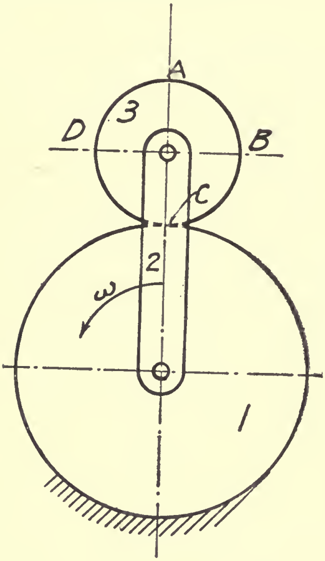


FIG. 123.

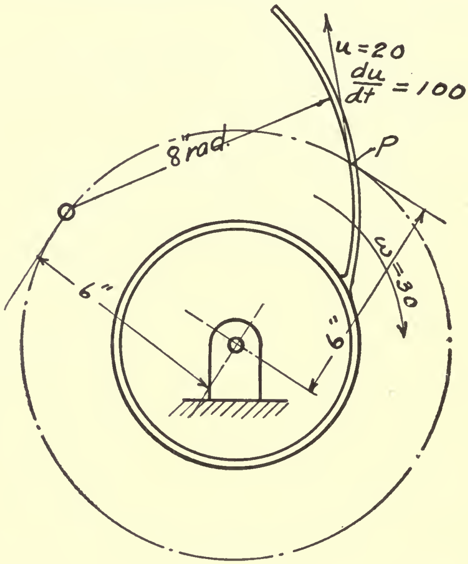


FIG. 124.

Find the accelerations of points A, B, C, and D on gear *3* by means of Coriolis' Law.

2. The vane A of a centrifugal pump, Fig. 124, rotates with uniform angular velocity 30 radians per second. Find the acceleration of a particle of water P, which moves along the vane with a relative velocity of 20 feet per second and relative acceleration 100 feet per second per second.

Radius of curvature of vane 8 inches, center of curvature 6 inches from center of rotation. Calculate each component of the acceleration and find the resultant graphically.

3. The flywheel of an engine, Fig. 125, rotates with angular velocity $\omega = 30$ radians per second, and angular acceleration $\alpha = 20$ radians per second per second. The governor weight is moving outward relative to the wheel with a uniform velocity of 10 feet per second. Using the dimensions given in Fig. 125, find the acceleration of G, the center of gravity of the governor.

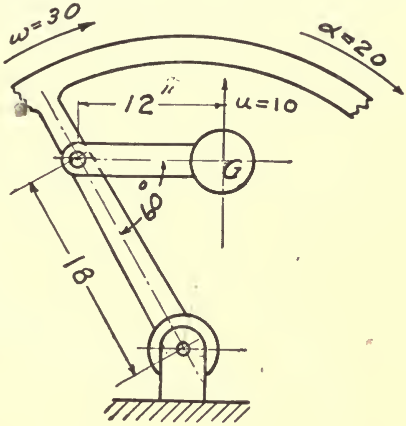


Fig. 125.

4. In Fig. 126 the bar link *2* has an angular velocity $\omega = 10$ radians per second and an unknown angular acceleration α. A particle P moves outward along the bar with relative velocity $u = 10$ feet per second, and unknown acceleration $\frac{du}{dt}$. The total acceleration of the point P is 200 feet per second per second directed horizontally to the left. Find the angular acceleration α of the bar and the acceleration of P along the bar $= \frac{du}{dt}$.

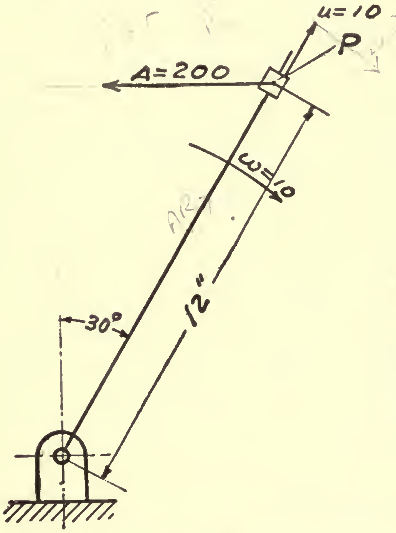


Fig. 126.

Hint.—The acceleration of P is made up of the following components:

(*a*) Normal acceleration of $P_2 = r\omega^2$;
(*b*) Tangential acceleration of $P_2 = r\alpha$;
(*c*) Acceleration along bar $= \frac{du}{dt}$;
(*d*) $2u\omega$.

Of these components (*a*) and (*d*) can readily be found, and the directions of (*b*) and (*c*) are known. Lay out the known components (*a*) and (*d*) as two sides of a polygon. Add two components in the direction of $r\alpha$ and $\frac{du}{dt}$ so that the closing side of the polygon will be the known resultant A.

69. Relative Accelerations.—The acceleration of a point A relative to a second point B is defined as the acceleration which added vectorially to the acceleration of B will give the acceleration of A. If the two points A and B, Fig. 127, are on the same

link, the total motion of A is composed of a translation with velocity and acceleration equal to the velocity and acceleration of B, together with a rotation about B with angular velocity ω and angular acceleration α. The acceleration due to the rotation is composed of two parts:

(a) A component $AB\omega^2 = \frac{V_{ab}^2}{AB}$ in the direction AB,

(b) A component $AB\alpha$ at right angles to AB.

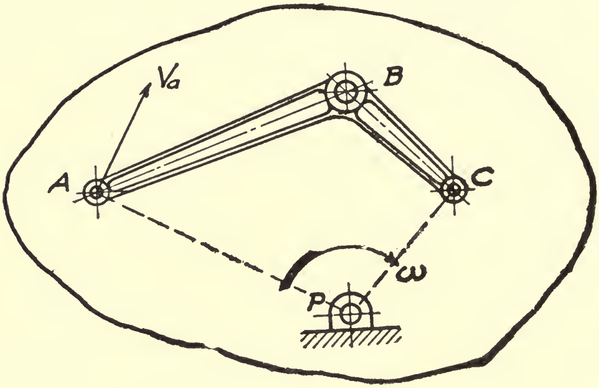


Fig. 127.

The component (a) is known as the *normal* acceleration of A relative to B, and the component (b) is known as the *tangential acceleration* of A relative to B.

70. Notation.—The following notation will be used to denote the various acceleration components:

A_a = Total acceleration of the point A.
A_{ba} = Acceleration of point B relative to point A.
A_a^n = Acceleration of A normal to the path of A.
A_a^t = Acceleration of A tangential to the path of A.
A_{ba}^n = Acceleration of B relative to A in the direction normal to V_{ba}.
A_{ba}^t = Acceleration of B relative to A in the direction parallel to V_{ba}.

In studying velocities a single equation of the form

$$V_b = V_a + V_{ba}$$

was often used. In studying accelerations a similar equation may be used, but it is usually necessary to resolve the accelerations into two components as follows:

$$A_b = A_b^n + A_b^t = A_a + A_{ba}^n + A_{ba}^t \text{ [1]}.$$

71. Graphical Constructions.—Terms of the form $\frac{V^2}{r}$ are used so frequently in determining accelerations that graphical constructions for finding such terms are very useful. Usually there is given a drawing of the link to a suitable scale, that is, r is represented by a fixed length on the paper. Two problems now arise:

(1) A vector representing the velocity may be given and the length of the acceleration vector $\frac{V^2}{r}$ required.

(2) The acceleration vector may be given and the length of the velocity vector required.

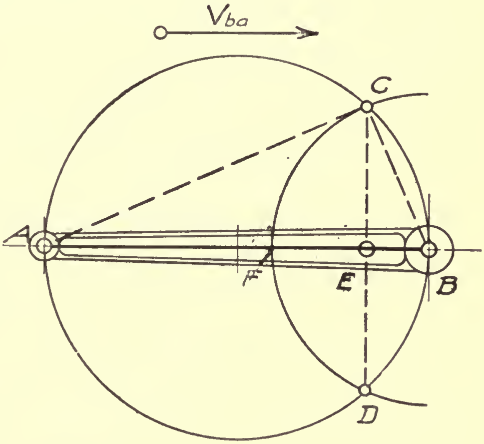


Fig. 128.

1. Let AB, Fig. 128, represent the radius r, and V_{ba} give the length of the velocity vector. On AB as a diameter draw a circle.

[1] Students are often puzzled in trying to determine whether to include the acceleration component $2u\omega$ in any given case, and whether to use the absolute or relative angular velocity of the link. A general rule covering these questions may be stated as follows:

"When the acceleration of a point on one link is to be determined by comparison with the acceleration of the *coincident* point on *another* link, *include* $2u\omega$ and use the *relative* angular velocity and acceleration of the two links in determining u and A_r. When the acceleration of a point is to be determined by comparison with the acceleration of *another* point on the *same* link, *omit* $2u\omega$, and use the *absolute* angular velocity and acceleration of the link in determining the *relative* acceleration of the two points."

With B as a center strike an arc CFD whose radius is V_{ba}. Draw CD intersecting AB at E. Then $BE=\frac{V_{ba}^2}{r}$.

Proof.—Triangles ABC and BCE are similar.
Therefore

$$\frac{BE}{CB}=\frac{CB}{AB}$$

or

$$BE=\frac{CB^2}{AB}=\frac{V_{ba}^2}{r}.$$

If $BC>AB$ this construction fails. A more general construction is shown in Fig. 129.

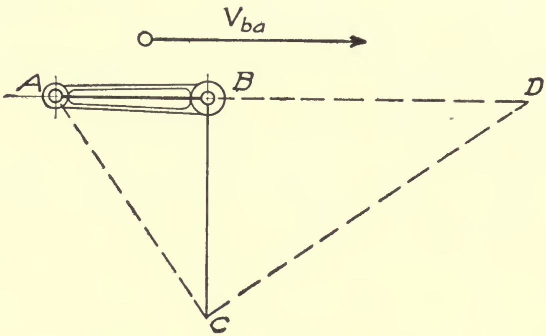


Fig. 129.

As before let $AB=r$ and the vector V_{ba} give the velocity of B relative to A.

At B erect a perpendicular BC of length V_{ba}. Join AC and through C draw CD perpendicular to AC intersecting AB extended at D. Then

$$BD=\frac{V_{ba}^2}{r}.$$

Proof.—Triangles ABC and CBD are similar.
Therefore

$$\frac{AB}{BC}=\frac{BC}{BD},$$

or

$$BD=\frac{BC^2}{AB}=\frac{V_{ab}^2}{r}.$$

2. Let AB, Fig. 130, represent the radius r and let AC be the acceleration vector. From C drop a perpendicular CD on AB.

Then

$$AD=\frac{V_{ba}^2}{r}.$$

On AB as a diameter draw a semicircle intersecting CD at E. Then $AE=V_{ba}$.

Proof.—Triangles AED and AEB are similar.

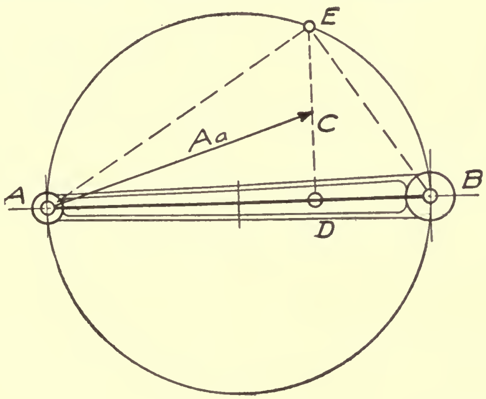


Fig. 130.

Therefore

$$\frac{AE}{AB}=\frac{AD}{AE}.$$

Therefore

$$AD=\frac{AE^2}{AB}=\frac{AE^2}{r}=\frac{V_{ba}^2}{r}.$$

Hence

$$AE=V_{ba}.$$

If $AD>AB$ this construction fails. In this case extend ABD to E so that $DE=AB$, Fig. 131.

With AE as a diameter describe a semicircle intersecting DC at F.

Then $DF=V_{ba}$.

Proof.—Triangles AFD and DFE are similar.

Therefore

$$\frac{DE}{DF}=\frac{DF}{AD},$$

or

$$AD=\frac{DF^2}{DE}=\frac{DF^2}{AB}=\frac{DF^2}{r}=\frac{V_{ba}{}^2}{r}.$$

Therefore

$$DF=V_{ba}.$$

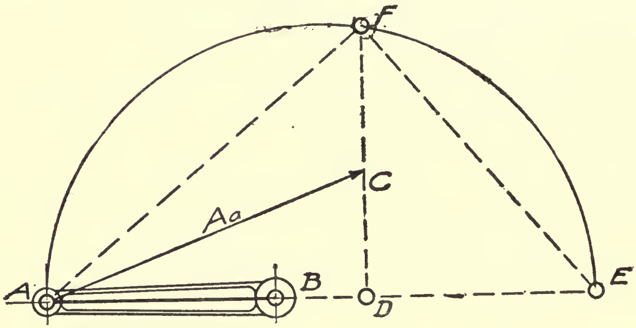


FIG. 131.

It should be observed that these constructions give the magnitude but not necessarily the directions of the vectors, V_{ba} or $\frac{V_{ba}{}^2}{r}$. The directions should be determined from the conditions of the problem.

72. Projection of Acceleration Components.—Let A_a and A_b, Fig. 132, be the accelerations of two points A and B connected

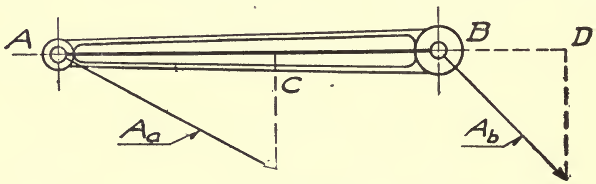


FIG. 132.

by a rigid link. These accelerations may be resolved into components along AB and at right angles to AB. Since:

$$A_a=A_b+A_{ba}{}^n+A_{ba}{}^t,$$

the components of A_a and A_b along the link must differ by $A_{ba}{}^n$, that is, by $\frac{V_{ba}{}^2}{AB}$. The component $A_{ba}{}^n$ must be in the direction BA. Therefore for the case shown A_a has a larger component along the link than A_b. If these components were reversed in direction A_b would have the larger component along the link.

73. Acceleration Image.—The acceleration of any point B, Fig. 133, on a moving link may be regarded as composed of the acceleration of any other point A, together with the acceleration due to the rotation of the link about A. This latter relative acceleration consists of two components:

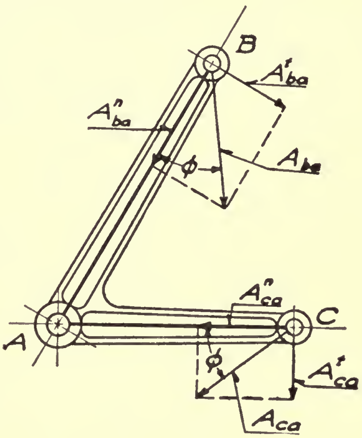


FIG. 133.

(1) The normal $A_{ba}{}^n = AB\omega^2$, which is directed from B toward A.

(2) The tangential $A_{ba}{}^t = AB\omega'$, where $\omega' = \frac{d\omega}{dt}$, at right angles to AB.

The resultant of these two is given by the equation:

$$A_{ba} = AB\sqrt{\omega^4 + \omega'^2}.$$

This resultant makes an angle ϕ with BA, where

$$\tan \phi = \frac{\omega'}{\omega^2}.$$

Similarly for another point C

$$A_{ca} = AC\sqrt{\omega^4 + \omega'^2},$$

and this resultant makes the angle ϕ with CA.

Therefore

$$\frac{A_{ca}}{A_{ba}} = \frac{CA}{BA}.$$

Now from a pole O, Fig. 134, lay off $Oa=A_a$, and lay off from a; $ac=A_{ca}$ and $ab=A_{ba}$. Then the triangle abc is similar to ABC because $\frac{ac}{ab}=\frac{AC}{AB}$ and the angle cab = angle CAB. Hence

$$Ob=A_b \quad \text{and} \quad Oc=A_c.$$

It follows, therefore, that if form a common pole O vectors be drawn representing the accelerations of the points of a rigid link, the ends of these vectors form an *image* of the original link. The sides of this image form an angle ϕ with the sides of the original link, where

$$\tan\phi=\frac{\omega'}{\omega^2}.$$

The acceleration image is of much importance in determining accelerations in mechanisms. If the accelerations of two points of a link are known, the accelerations of all other points can be found by simply constructing a figure similar to the link.

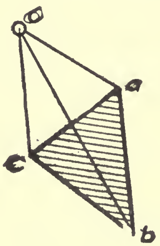


Fig. 134.

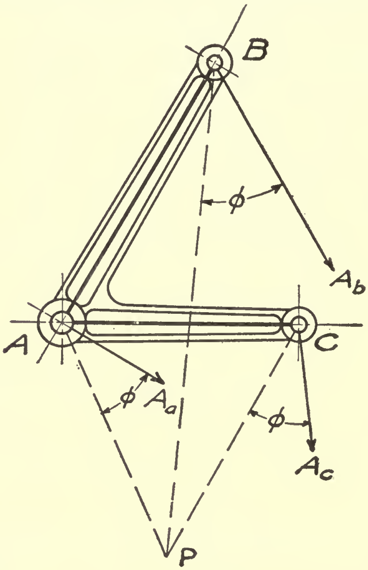


Fig. 135.

74. Acceleration Center.—Each point of the acceleration diagram gives the acceleration of a similar point on the link. The pole O is therefore the image of a point of zero acceleration on the original link. This point is called the *center of acceleration*. It is found by constructing the quadrilateral $ABCP$, Fig. 135, similar to $abcO$. The lines PA, PB, etc., make the angle ϕ with A_a, A_b, etc. The accelerations A_a, A_b, etc., are also proportional to PA,

PB, etc. That is, the link behaves, as far as acceleration is concerned, as if it were rotating about a fixed center at *P*.

The center of acceleration and the center of velocity are quite distinct points.

The center of acceleration is not practically of much importance. In the first place there is no simple manner of determining this center, such as was furnished by the law of three centers in determining centers of velocity. Secondly, the location of the center of acceleration depends not only on the configuration of the mechanism, but also on the accelerations themselves. A simple illustration will make this clear. Consider a wheel, Fig. 136, rolling at uniform speed along a rail. The center of the wheel has no acceleration and is consequently the center of acceleration. On the other hand, if the speed of the wheel is changing the center is being accelerated. Therefore, the center of acceleration must now be at some other point. The locus of the center of acceleration is discussed in *note C*.

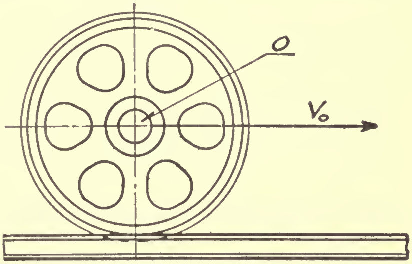


Fig. 136.

75. Graphical Methods.—The accelerations in mechanisms may be determined by means of the method of relative accelerations, by Coriolis' law or by a combination of these two. Where possible, the method of relative accelerations is usually preferred. In some cases this method fails and Coriolis' law must be applied.

76. The Acceleration Polygon.—If from a common pole vectors are drawn, representing the accelerations of different points of a mechanism, each link will be represented by an image. The complete figure showing the accelerations of all points is called an *acceleration polygon* for the mechanism. In general, in the mechanisms for which the velocity polygons are most easily constructed, the acceleration polygons are also relatively simple; and those in which special constructions must be used to form the velocity polygon require more elaborate devices to determine the accelerations.

In determining the accelerations, components of the form $\frac{V^2}{r}$ occur so frequently that it is best to start, in most cases, by constructing a velocity polygon. This gives at once, all the relative velocities which may be required.

77. Selection of Scale.—In determining the proper scale for constructing this polygon, the principles of Art. 71 should be followed. Thus if the acceleration of any point is given, its normal component is $\frac{V^2}{r}$. When this is laid off, the vector representing the velocity is first found, and then the velocity polygon is constructed to the scale thus determined.

One case deserves special attention. Suppose some link to be rotating at constant speed about a fixed center. The acceleration of any point is normal and is equal to $\frac{V^2}{r}$. Now if the scale of accelerations be so chosen that the radius itself represents the acceleration.

Then

$$\frac{V^2}{r}=r.$$

Therefore $V=r$.

That is, the radius also represents the velocity to scale. This scale is often used in such cases.

78. Ordinary Gear Train.—In Fig. 137, let gear 2 rotate with a known angular velocity and known acceleration.

It is required to find the acceleration of gears 3 and 4. From a pole O, Fig. 138, lay off Of = normal acceleration of $A=R_2\omega_2{}^2$, and from f lay off fa = tangential acceleration of $A=R_2\omega'_2$.

Then the acceleration image of gear 2 is a circle about O as a center with radius Oa. Locate the image of B by constructing angle $aOb=AP_2B$. The acceleration of B can be resolved into normal and tangential components. Since the speed of C must always be equal to that of B the tangential component of A_c is the same as that of A_b. The normal component, however, is quite different.

$$A_c{}^n=\frac{V_c{}^2}{r_3}=\frac{V_b{}^2}{r_3}=\frac{V_b{}^2}{r_2}\times\frac{r_2}{r_3}=A_b{}^n\frac{r_2}{r_3}.$$

That is, the normal accelerations are inversely proportional to the radii. Also A_c^n is opposite in direction to A_b^n. Lay off $Og = A_c^n$ parallel to CP_3. From g draw $gc = fa$ perpendicular to Og. Then Oc = acceleration of C. The image of the compound gear 3 is a pair of circles with centers at O. The larger has a radius Oc, and the smaller radius is found by proportion. The image of D is found by constructing angle $cOd = CP_3D$. The image of E is found in exactly the same manner as that of C. The polygon is thus a system of concentric circles.

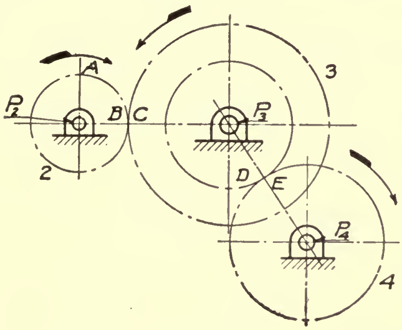


FIG. 137.

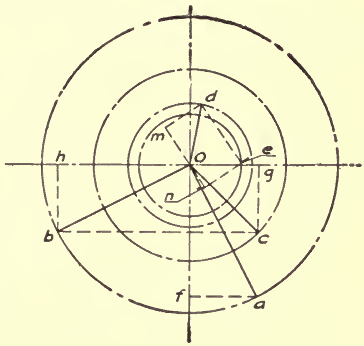


FIG. 138.

79. Epicyclic Gear Train.—In the reverted train, Fig. 139, wheel 1 is stationary. Arm 2 revolves about O with angular velocity ω and acceleration $\dfrac{d\omega}{dt}$. The two wheels 3 are carried by the pin at A and revolve together. The gear 4 is free to turn about O.

The acceleration of A is composed of two parts:

$$A_a^n = l\omega^2,$$

$$A_a^t = l\frac{d\omega}{dt}.$$

$$A_b = A_a + A_{ba}^n + A_{ba}^t.$$

$$A_{ba}^n = \left(\frac{l}{r}\omega\right)^2 r = \frac{l^2\omega^2}{r},$$

since

$$\omega_3 = \frac{l}{r}\omega.$$

As B always moves twice as fast as A,

$$A_{ba}{}^t = A_a{}^t = l\frac{d\omega}{dt}.$$

From the pole O, Fig. 140, lay off $Om = l\omega^2$ and from m draw $ma = l\frac{d\omega}{dt}$. Then $Oa = A_a$.

From a lay off $an = \frac{l}{r}(Om)$ and from n draw $nb = l\frac{d\omega}{dt}$. Then $Ob = A_b$.

The acceleration image of the pair of gears 3 is a pair of circles having a as a center. The radius of the inner circle is ab. The radius of the outer one is found by proportion. The accelerations of C and D are found by simply drawing the diameter $bacd$. As may be readily shown the acceleration of C is entirely along the radius CA. D has a tangential acceleration dp which is readily seen to be equal to

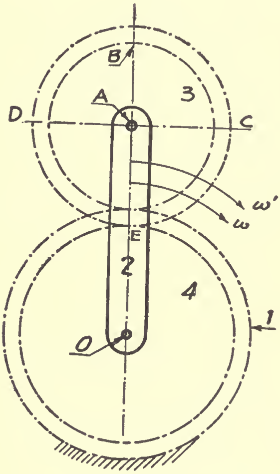


FIG. 139.

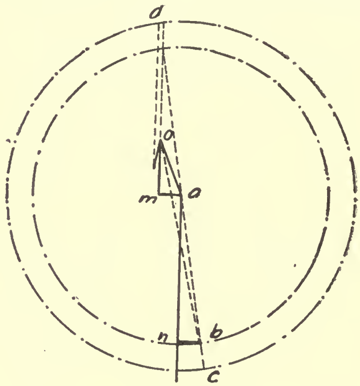


FIG. 140.

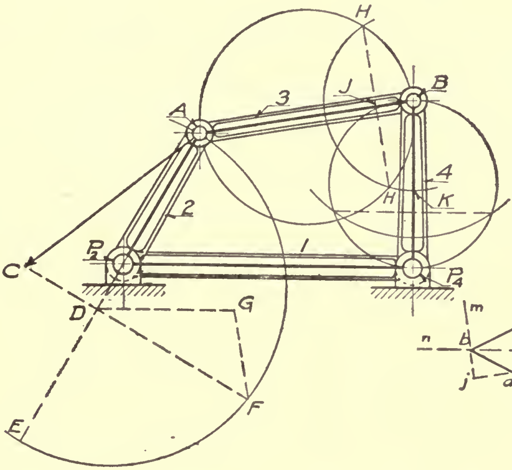
$$am\frac{cd}{ac} = l\frac{d\omega}{dt}\cdot\frac{CD}{AC}.$$

The acceleration of E has two components: (1) a tangential $= dp$ and (2) a normal $= \frac{V_e{}^2}{OE} = \frac{V_d{}^2}{OE}$, since D and E have the same velocity. Since the wheel 3 revolves about C,

$$V_d = V_a\cdot\frac{CD}{CA}.$$

From these data $\frac{V_e^2}{OE}$ can be found either by calculation or by a graphical method, the details of which are left to the student. The same results may be obtained by purely analytical methods or by use of Coriolis' Law.

80. The Four-link Chain.—In the four-link chain, Fig. 141, let AC represent the acceleration of A. It is required to find the acceleration of B. From C drop a perpendicular CD on AP.



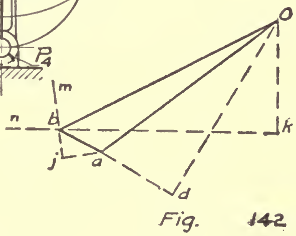


FIG. 141. FIG. 142.

Then

$$AD = A_a^n = \frac{V_a^2}{r}.$$

To find V_a prolong AP_2D to E so that $DE = AP_2$. On AE construct a semicircle intersecting CD at F.

Then

$$DF = V_a. \quad \text{(Art 71.)}$$

Complete the velocity polygon by drawing DG perpendicular to P_4B and FG perpendicular to AB.

Then

$$DG = V_b \quad \text{and} \quad FG = V_{ba}.$$

From a pole O, Fig. 142, lay off $Oa = AC =$ acceleration of A.

$$A_b = A_a + A_{ba}^n + A_{ba}^t.$$

$$A_{ba}^n = \frac{V_{ba}^2}{AB}.$$

The normal component is found by drawing a circle on AB as a diameter, and striking an arc with B as center and radius $FG = V_{ba}$. Then

$$BJ = \frac{V_{ba}^2}{AB}. \quad \text{(Art 71.)}$$

This acceleration is directed from B toward A. Lay off $aj = BJ$ in the direction BA. The third component A_{ba}^t is known to be perpendicular to AB. Through j draw a line jm in this direction. Then the acceleration image of B lies on jm. It is also known that since B rotates about P_4 its acceleration is made up of two components as given by the equation

$$A_b = A_b^n + A_b^t,$$

where

$$A_b^n = \frac{V_b^2}{P_4B}.$$

A_b^n is directed from B toward P_4 and is represented by BK. From O lay off $Ok = BK$. The other component A_b^t is perpendicular to P_4B. Through k draw a line kn perpendicular to P_4B. Then the image of B lies on kn. This locates the image of B at b. Connecting ab and Ob it is readily seen that Oa is the image of link *2*, Ob the image of link *4* and ab the image of link *3*. The angular accelerations of the links are easily found from this figure. Thus:

$$ad = CD = P_2A\,\omega'_2,$$
$$bj = AB\omega'_3,$$
$$kb = P_4B\omega'_4.$$

EXERCISE

5. Construct the acceleration polygon for the crossed quadric chain, Fig. 60. Crank 2 rotates uniformly. Find the acceleration of the center 24:

(*a*) Considered as a point on link *2*.
(*b*) Considered as a point on link *4*.

81. The Steam Engine Mechanism.—In the steam engine mechanism, Fig. 143, let AC represent the acceleration of the crank pin A. It is required to find the acceleration of the wrist pin B. The velocities of the crank pin, of the wrist pin and the

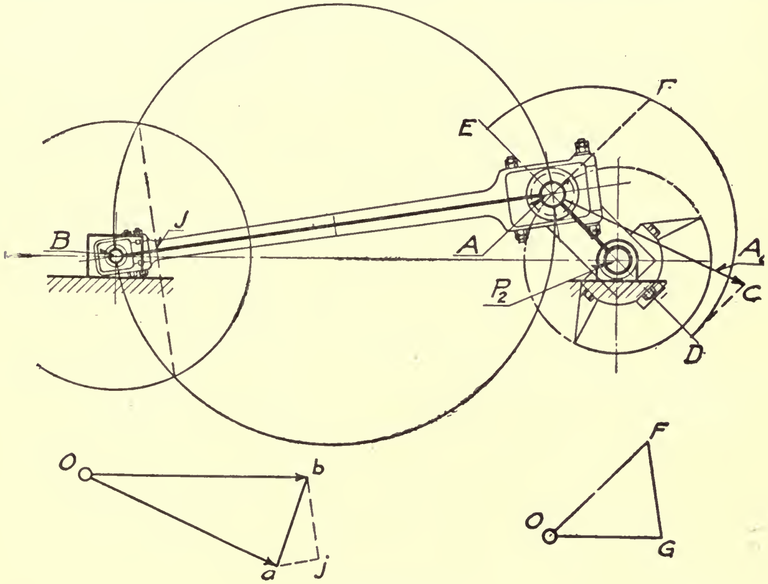


FIG. 143.

relative velocity of the two are found precisely as in the preceding case.

$$AF = OF = V_a,$$

$$OG = V_b,$$

$$FG = V_{ba}.$$

$$A_b = A_a + \frac{V_{ba}^2}{AB} + A_{ba}^t.$$

BJ found precisely as in the preceding paragraph represents the component $\frac{V_{ba}^2}{AB}$. From a pole O lay out $Oa = AC = A_a$. From a draw $aj = BJ = A_{ba}^n$ parallel to BA. The third component A_{ba}^t is perpendicular to AB. Draw a line jb in this direction. Then the image of B lies on this line. The acceleration of B is

in a horizontal direction. Therefore a horizontal line through O also contains the image of B. This locates b. The acceleration of B is Ob. ab is the image of the rod.

This construction differs from that for the four-link chain only in the fact that B has no acceleration normal to its path.

82. Klein's Construction.—The case where the crank rotates at uniform speed occurs so frequently that it deserves special consideration.

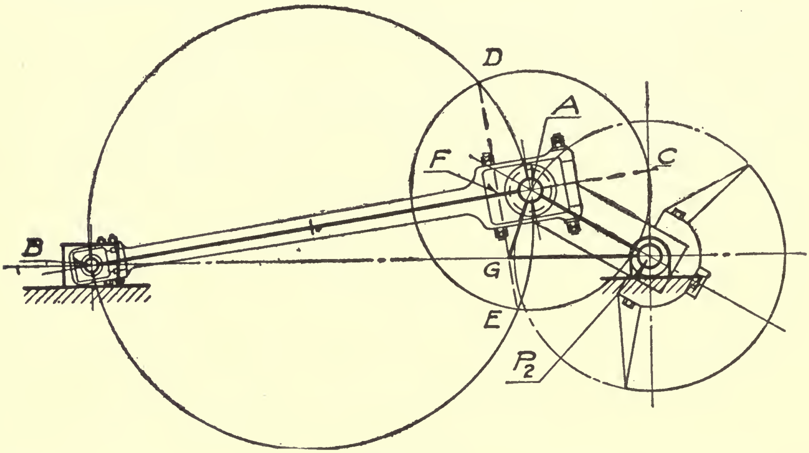


FIG. 144.

In this case it has been shown (Art. 77) that the crank radius itself can be used to represent both the acceleration and the revolved velocity of the crank pin A. Let P_2, Fig. 144, be regarded as the pole for both accelerations and velocities. Then

$$V_a = P_2A,$$
$$V_b = P_2C,$$
$$V_{ba} = CA,$$
$$P_2A = A_a \text{ reversed in direction.}$$

On BA as a diameter draw a circle. Around A as a center draw a circle of radius AC, intersecting the first circle at D and E. Draw DE, cutting BA at F and BP_2 at G. Then

$$AF = \frac{V_{ba}^2}{AB} = A_{ba}^n,$$

reversed in direction

$$A_b = A_a + A_{ba}^n + A_{ba}^t.$$

The first two of these components are known, and the third must be in a direction perpendicular to AB, that is, in the direction FE. The resultant acceleration of B must be in a horizontal direction. Therefore P_2G represents the acceleration of B reversed in direction. AG is the image of the rod AB. This is known as *Klein's Construction.*

83. Sliding Pairs.—In the four-link mechanism shown in Fig. 145, the block, link *4*, moves in a circular slot in link *3*. It was shown in Art. 5 that such a block is equivalent to a link BA, whose length is the radius of curvature of the slot, and which is joined to link *3* at B, the center of curvature. The mechanism is therefore equivalent to the ordinary four-link chain P_3BAP_2. If the acceleration of A is known, the acceleration of B is readily found, and therefore the acceleration of any other point on link *3* is easily determined. If, however, the slot is made straight, the center B recedes to infinity, and the construction fails. To determine accelerations in this case Coriolis' Law must be applied.

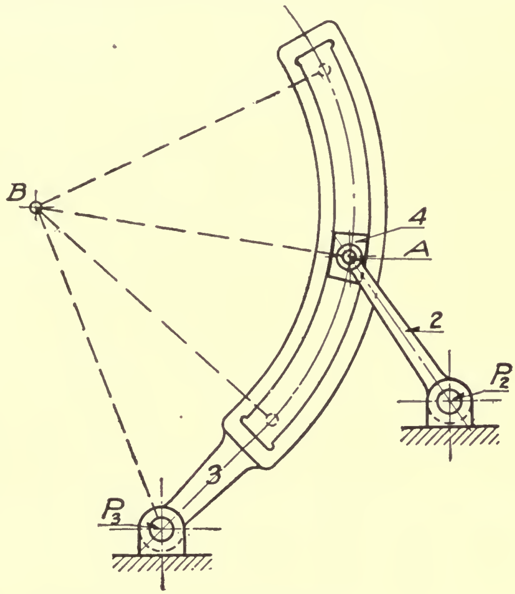


Fig. 145.

In the shaper mechanism, Fig. 146, the acceleration of A is known. It is required to find the acceleration of B. Let A_4 be a point on link *4* directly behind A. Then by Coriolis' Law,

$$A_a = A_{a_4} + A_r + 2u\omega,$$

where A_r is the acceleration due to the sliding of link *3* along *4*, u is the velocity of this sliding and ω is the angular velocity of

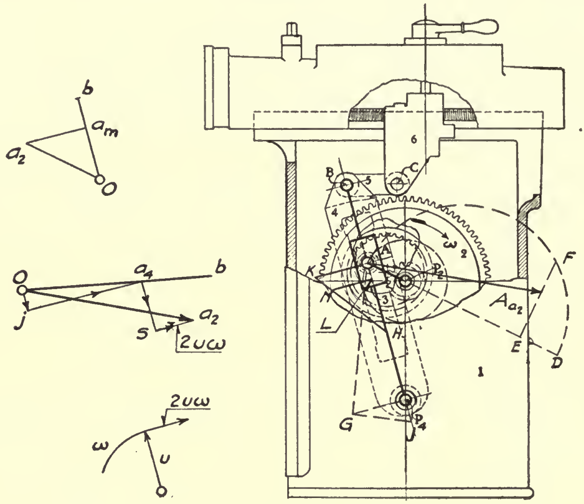


FIG. 146.

link *4* about P_4. In the usual way find the velocity of $A = EF$, and complete the velocity polygon as shown. Then,

$$Oa_m = V_{a_4} \quad \text{and} \quad a_2a_m = u.$$

$$A_{a_4} = A_{a_4}{}^n + A_{a_4}{}^t,$$

$$A_{a_4}{}^n = \frac{V_{a_4}{}^2}{P_4A} = P_4J,$$

$$\omega = \frac{V_{a_4}}{P_4A},$$

$$2\omega = \frac{V_a}{\frac{1}{2}P_4A} = \frac{Oa_m}{\frac{1}{2}P_4A}.$$

From A lay off $AK = Oa_m$ perpendicular to P_4A. Draw a line from H, the middle point of P_4A to K. Then

$$\tan AHK = \frac{Oa_m}{\frac{1}{2}P_4A} = 2\omega.$$

From H lay off $HL = a_2a_m = u$. Through L draw a line parallel to AK intersecting HK at N. Then $LN = HL \tan AHK = 2u\omega$. If the rotation is clockwise, u is directed upward along P_4A. The usual rule, therefore, shows that $2u\omega$ is perpendicular to P_4A, and is directed to the right. Of the four components of $A_a(A_{a_4}{}^n, A_{a_4}{}^t, A_r$ and $2u\omega)$ two are now completely known. A_r is along P_4A and $A_{a_4}^t$ is perpendicular to P_4A. This gives the directions of the remaining two components. Since the resultant A_a is known, this suffices to construct the acceleration polygon.

From a pole O lay off $Oa_2 = A_a$ and $Oj = A_{a_4}{}^n$. From a_2 lay off $a_2s = 2u\omega$ reversed in direction. From j draw a line perpendicular to P_4A, and from s draw a line parallel to P_4A. The intersection of these lines is the image of A_4. $Oa_4 = A_{a_4}$. To find the acceleration of b prolong Oa_4 to b so that

$$\frac{Oa_4}{Ob} = \frac{P_4A}{P_4B}.$$

Study this construction carefully. Note that $Oa_2 = A_a$ is the resultant of four components:

(1) $Oj = A_{a_4}{}^n$.
(2) $ja_4 = A_{a_4}{}^t$.
(3) $a_4s = A_r$.
(4) $sa_2 = 2u\omega$.

EXERCISE

6. An 8×12 inch engine runs uniformly at 300 r.p.m. Length of connecting rod $= 42$ inches. Find the acceleration of the piston when the crank is 30° from the head end dead center.

Use Klein's Construction. Check the results by Coriolis' Law.

84. Blake Stone Crusher.—The principles developed in the preceding paragraphs are sufficient to find accelerations in all mechanisms where the velocity polygon can be constructed without resort to special devices such as the three-line or four-line constructions. As an example, consider the Blake stone crusher, Fig. 147.

Let AL = acceleration of A. It is required to find the accelerations of B, C, D and E. Determine the scale of velocities and construct the velocity polygon as shown.

The acceleration polygon is constructed as follows: From a pole O, lay out $Oa = AL =$ acceleration of A. The acceleration of B is made up of three components $A_b = A_a + A_{ba}^n + A_{ba}^t$. Of these A_a is known, $A_{ba}^n = \dfrac{V_{ba}^2}{AB}$ and is in the direction BA, and

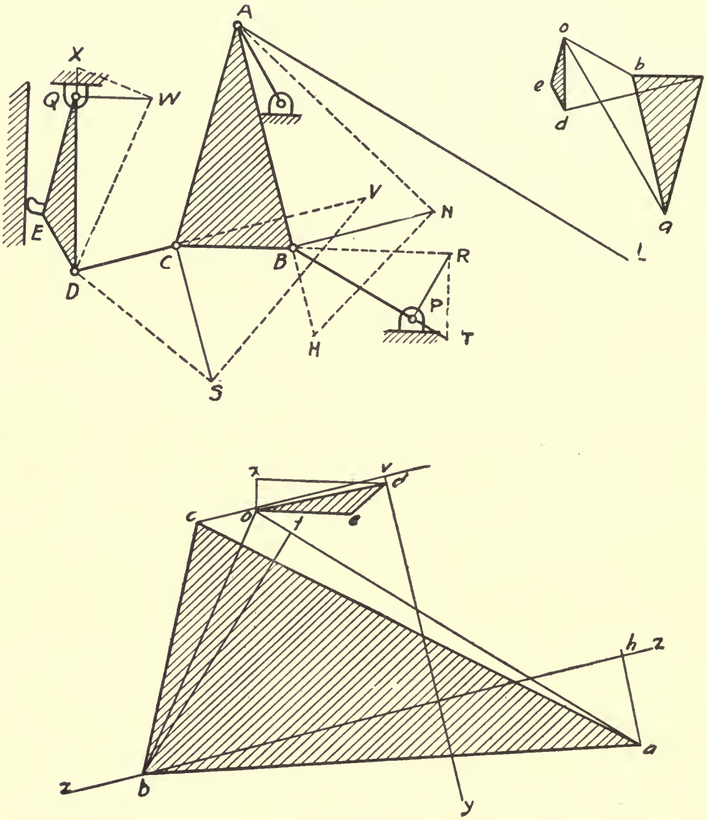


Fig. 147.

A_{ba}^t is at right angles to BA. To find A_{ba}^n lay off from B the velocity $V_{ba} = BN$ (taken from the velocity polygon) at right angles to BA, join AN and draw NH at right angles to AN. Then $BH = \dfrac{BN^2}{AB} = A_{ba}^n$. Lay off $ah = BH$ from a in the direction BA and draw hz perpendicular to AB. Now the image of B lies on

hz. The acceleration of B may also be regarded as composed of two parts, $A_b = A_b^n + A_b^t$, along BP and perpendicular to BP respectively.

$$A_b^n = \frac{V_b^2}{PB}.$$

From P lay off $PR = V_b$ (taken from the velocity polygon) at right angles to PB. Join BR and draw RT perpendicular to BR. Then $PT = A_b^n$. From the pole O lay off $Ot = PT$ and draw tb perpendicular to Ot. The point b where this line intersects hz is the image of B, and $Ob = A_b$. Join ab. Then ab is the acceleration image of AB. The acceleration of C is readily found by constructing triangle abc similar to ABC. Then $Oc = A_c$.

A similar process gives the acceleration of D and E. $A_d = A_c + A_{dc}^n + A_{dc}^t$. A_c is known. $A_{dc}^n = \frac{V_{dc}^2}{DC}$ and is the in direction DC. A_{dc}^t is at right angles to DC.

To find A_{dc}^n draw from C the line $CS = V_{dc}$ (taken from the velocity polygon) at right angles to DC. Join DS and draw SV perpendicular to DS. Then:

$$CV = A_{dc}^n.$$

Lay off $cv = VC$ and draw vy perpendicular to cv. Then the image of D lies on vy.

The acceleration of D may also be regarded as composed of two parts,

$$A_d = A_d^n + A_d^t,$$

along DQ and perpendicular to DQ respectively.

$$A_d^n = \frac{V_d^2}{QD}.$$

From Q lay off $QW = V_d$ (taken from the velocity polygon) at right angles to QD. Join DW and draw WX perpendicular to DW. Then:

$$A_d^n = QX.$$

From O lay off $Ox = QX$ and draw xd perpendicular to Ox. The point d where this line intersects vy gives the acceleration of $D = Od$. Join cd. The acceleration of E is found by constructing

a triangle *Ode* similar to *QDE* and the polygon is now complete. Note that *Oa* is the image of link *2*, *abc* the image of link *3*, *Ob* the image of link *4*, *cd* the image of link *5* and *Ode* the image of link *6*.

EXERCISES

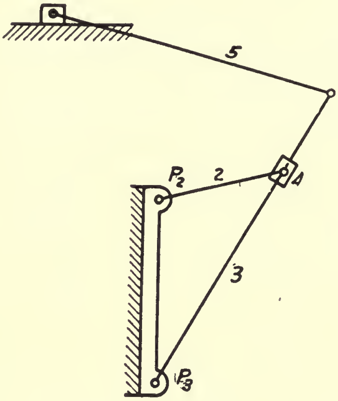


FIG. 148.

7. Construct acceleration polygon for the quick-return motion, Fig. 112. Gear 2 rotates uniformly.

8. Construct acceleration polygon for the Joy valve gear, Fig. 94. The crank, link *2*, rotates uniformly.

9. Construct acceleration polygon for the shaper mechanism, Fig. 148. The crank rotates uniformly.

10. Construct acceleration polygon for the Pilgrim step motion, Fig. 96.

Hint.—The accelerations of the pitch points of two gears in mesh have the same components along the tangents to the pitch circles, but different components along the line of centers.

After completing these polygons draw vectors from each point whose acceleration has been determined, showing these accelerations in direction and magnitude. This will serve to give the student a better physical conception of the meaning of the polygon.

85. Three-line Construction.—For mechanisms in which the velocity polygon is found by the three-line construction, Art. 56, the acceleration polygon is found by an analogous method. Example: The Stephenson link motion, Fig. 149. Let link *2* revolve at constant speed and let *Oa* of the velocity polygon represent the revolved velocity of *A*. The velocity polygon is constructed by the method of Art. 56. The accelerations of *A* and *B* are represented by *Oa* and *Ob* of the acceleration polygon. The ordinary methods of finding accelerations of other points fail and it is necessary to adopt a new method. Consider the link *5* extended so as to include the point *M*. Then:

$$A_m = A_c + A_{mc}^n + A_{mc}^t,$$
$$= A_d + A_{md}^n + A_{md}^t,$$
$$A_c = A_a + A_{ca}^n + A_{ca}^t,$$
$$A_d = A_d^n + A_d^t.$$

Then:

$$A_m = A_a + A_{ca}^n + A_{ca}^t + A_{mc}^n + A_{mc}^t.$$

Of these A_a is known,

$$A_{ca}^n = \frac{V_{ca}^2}{AC}, \quad A_{mc}^n = \frac{V_{mc}^2}{MC}.$$

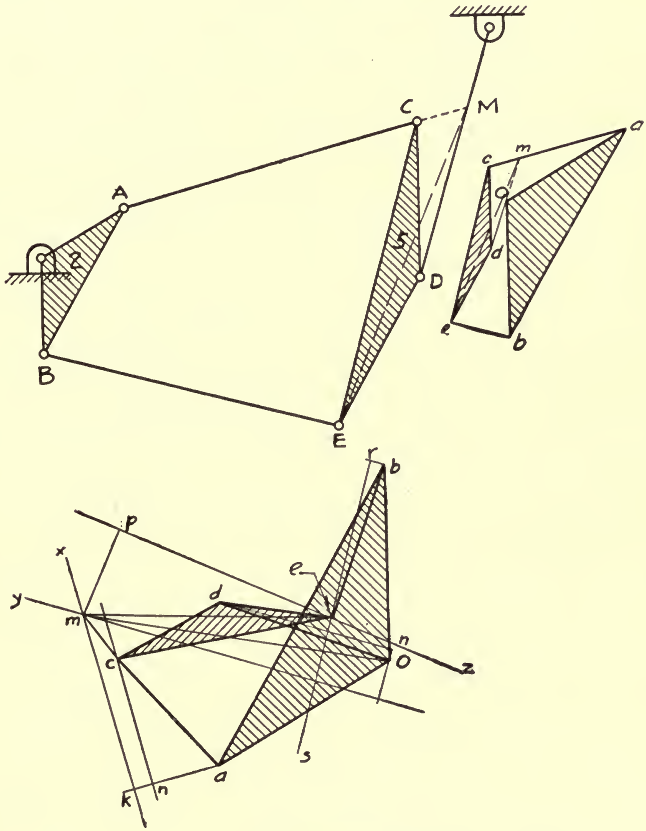


FIG. 149.

The velocities V_{ca} and V_{mc} can be taken from the velocity polygon. The components A_{ca}^n and A_{mc}^n are thus found. They lie in the direction CA and MC, repsectively, and therefore can be added directly. The components A_{ca}^t and A_{mc}^t are both at right angles to ACM. The acceleration of M is therefore composed

of three known quantities and a component whose direction is known.

Also:

$$A_m = A_d^n + A_d^t + A_{md}^n + A_{md}^t,$$

$$A_d^n = \frac{V_d^2}{PD},$$

in the direction of DP,

$$A_{md}^n = \frac{V_{md}^2}{MD},$$

in the direction MD, and the other components are both at right angles to PMD.

We can now find the acceleration of M as shown in the acceleration polygon. From the pole O lay off $Oa = A_a$. From a lay off $ah = A_{ca}^n$ and from h lay off $hk = A_{mc}^n$. Draw kx perpendicular to ak. Then the acceleration image of M lies on kx.

From O lay off $On = A_d^n$ and from n lay off $nl = A_{md}^n$. Through l draw ly perpendicular to On. Then the acceleration image of M lies at m, the intersection of ly and kx.

To find the acceleration of a second point E on link *5* consider the following equations:

$$A_e = A_m + A_{em}^n + A_{em}^t = A_b + A_{eb}^n + A_{eb}^t.$$

A_m is known $= Om$. $A_{em}^n = \frac{V_{em}^2}{EM}$ in the direction EM, and A_{em}^t is at right angles to EM.

From m lay off $mp = \frac{V_{em}^2}{EM}$ and draw pz perpendicular to mp. Then the acceleration image of E lies on pz. A_b is also known $= O_b$. $A_{eb}^n = \frac{V_{eb}^2}{EB}$ in the direction EB. A_{eb}^t is perpendicular to EB.

From b lay off $br = \frac{V_{eb}^2}{EB}$ and draw rs perpendicular to br. Then the acceleration image of E is at e, the intersection of pz and rs.

Join me. Then to find the accelerations of C and D it is necessary only to construct a figure $mced$ similar to $MCED$. To check the accuracy of the work it should be noted that d should lie on a line through n perpendicular to PD, and c on a line through h perpendicular to AC.

86. Four-line Construction.—In Art. 59 it was seen that the velocity images of two links can be constructed if loci are known for the images of two points on each link, and provided that the instantaneous center of relative motion can be found. In constructing the acceleration polygon for such a mechanism two cases may arise:

(1) There may be a joint between the two links.

(2) There may be no joint between the two links.

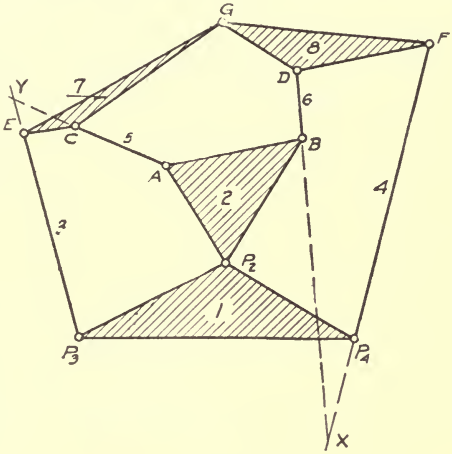


FIG. 150.

As an example of the first case, consider the eight-link mechanism shown in Fig. 150. Let link *1* be the stationary link, and let the velocity and acceleration of *A* on link *2* be known. Let *Oa*, Fig. 151 represent the velocity of *A*.

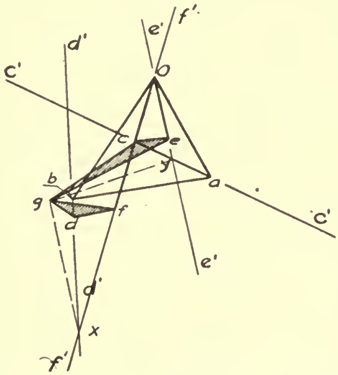


FIG. 151.

The velocity of *B* is first found, and a velocity image of link *2* is constructed as *Oab*. Then *c'ac'* parallel to *AC* is a locus of the image of *C* on link *7*, and *e'Oe'* parallel to P_3E is a locus of the image of *E*. Similarly *d'bd'* and *f'Of'* are loci for the images of *D* and *F* on link *8*. The instantaneous center *78* is the point *G*. To construct the velocity polygon imagine link *7* extended to include *Y* the intersection of *AC* and P_3E prolonged. Then *y* the intersection of *c'ac'* and *e'Oe'* is the image of *Y* on link *7*. Similarly *X* is located on link *8* by the intersection of *BD* and P_4F. Then the intersection of *d'bd'* and *f'Of'* is the image of *X*. Join *XG*

and YG. Draw xg and yg parallel to XG and YG respectively. Then the intersection of xg and yg is the image of G. The images of E, C, D, and F are now found by drawing lines from g parallel to GE, GC, GD and GF. The polygon is now complete.

To construct the acceleration polygon start with Oab, Fig. 152, as the acceleration image of link *2*. Next find the acceler-

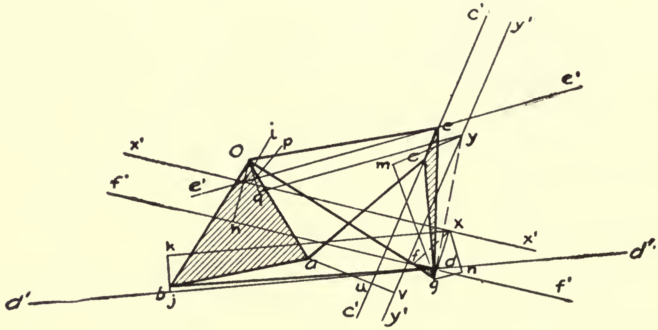


FIG. 152.

ation images of X and Y. To get the acceleration image of X write the equation

$$A_x = A_f + A_{xf} = A_f^n + A_f^t + A_{xf}^n + A_{xf}^t.$$

Of these four components A_f^n and A_{xf}^n are readily found. From O lay off $Oh = A_f^n = \frac{V_f^2}{P_4F}$, parallel to P_4F. Through h draw $f'hf'$ perpendicular to P_4F. Then $f'hf'$ is a locus for the image of F. From h lay off $hi = A_{xf}^n = \frac{V_{xf}^2}{XF}$ parallel to XF. Oi then represents $A_f^n + A_{fx}^n$. The remaining components A_f^t and A_{xf}^t are both perpendicular to P_4F. Therefore a line $x'x'$ through i perpendicular to P_4F gives a locus for the image of X.

Next consider the equation

$$A_x = A_d + A_{xd} = A_b + A_{bd} + A_{xd} = A_b + A_{db}^n + A_{db}^t + A_{xd}^n + A_{xd}^t.$$

Of these five components, three are known:

$$A_b = Ob,$$

$$A_{db}^n = \frac{V_{db}^2}{DB},$$

$$A_{xd}^n = \frac{V_{xd}^2}{XD}.$$

From b lay off $bj = \frac{V_{db}^2}{DB}$ parallel to DB. Through j draw $d'jd'$ perpendicular to BD, giving a locus for the image of D. From j lay off $jk = \frac{V_{xd}^2}{XD}$ parallel to XD. The two components A_{db}^t and A_{xd}^t are both perpendicular to BD. Through k draw a line perpendicular to BD, giving a second locus for the image of X. The intersection of the two loci gives x as the image of X. In the same manner the image of Y is located at y. Next to find the image of G consider the equations

$$A_g = A_x + A_{gx}^n + A_{gx}^t$$

$$= A_y + A_{gy}^n + A_{gy}^t.$$

Of these components A_x and A_y are known, $A_{gx}^n = \frac{V_{gx}^2}{GX}$, and $A_{gy}^n = \frac{V_{gy}^2}{GY}$. From x lay off $xn = \frac{V_{gx}^2}{GX}$ parallel to Gx and through n draw a line perpendicular to GX. This line is a locus of the image of G. Similarly from y lay off $ym = \frac{V_{gy}^2}{GY}$ parallel to GY, and through m draw a line perpendicular to GY. This line is a second locus for the image of G. The intersection of the two loci gives g, the image of G.

To complete the figure $gxfd$ similar to $GXFD$, and $gyec$ similar to $GYEC$. Note that there was already one locus for each of the points f, d, e, and c. If the work is correctly done these points should fall on the proper loci.

EXERCISE

11. Construct an acceleration polygon for the Walschaert valve gear, Fig. 110. The crank, link *2*, rotates uniformly.

CASE 2. The acceleration polygon for this case cannot be constructed directly by any of the methods so far considered.

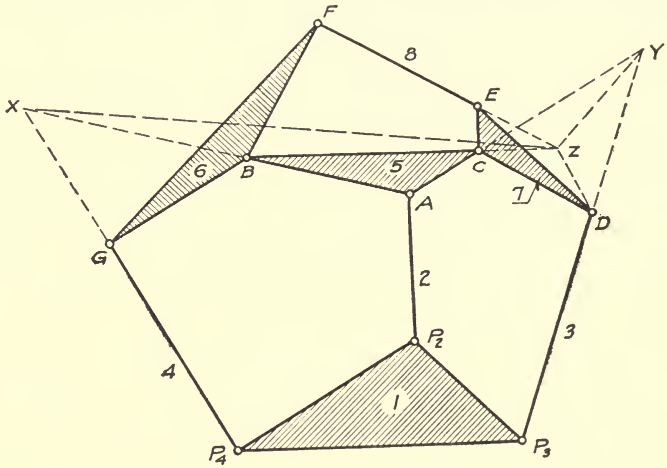


FIG. 153.

The accelerations may be determined in some instances by means of the trial and error method described below.

Consider the eight-link mechanism shown in Fig. 153. The velocity polygon, Fig. 154, is constructed in the usual manner.

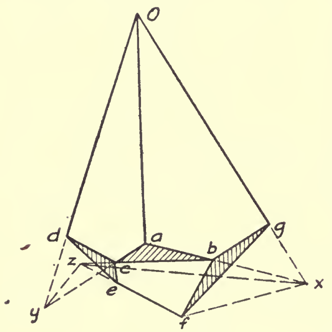


FIG. 154.

Starting with the acceleration of A, Fig. 155, the accelerations of X and Y are found precisely as in the preceding article. In finding the images of X and Y one locus is determined for the image of each of the points B, C, D and G. Since X is considered as a point on link *6* we may write

$$A_f = A_x + \frac{V_{fx}^2}{FX} + A_{fx}^t,$$

which gives a locus for the image of F.

Also since Y is a point on link *7*

$$A_e = A_y + \frac{V_{ey}^2}{EY} + A_{ey}^t,$$

which gives a locus for the image of E.

As there is only one locus for the image of each point, it is impossible to find the image of any point directly. Choose some point b_1 on the locus of b and assume temporarily that this is the image of B. Then since link *2* is represented by a similar image abc, the image of C is readily found at c_1. Also since link *6* is represented by a similar image, f_1 the image of F is found by con-

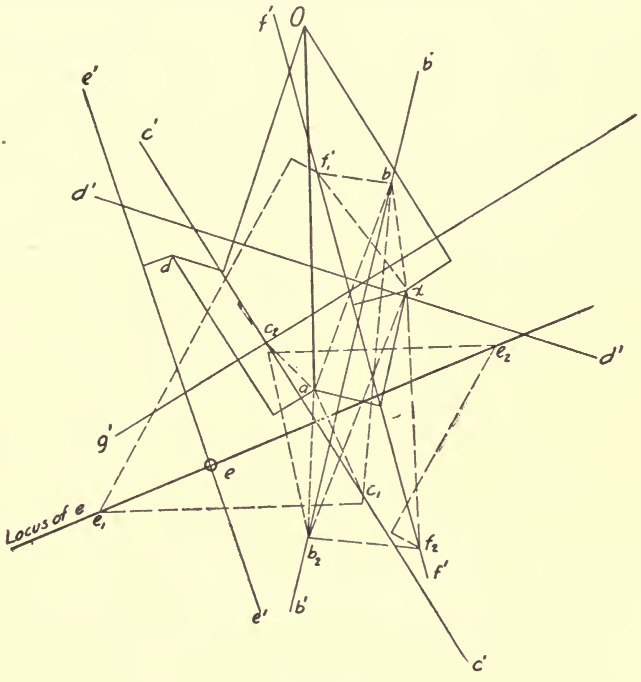


FIG. 155.

structing triangle xb_1f_1 similar to XBF. Now e_1, the image of E is determined by the equations

$$A_{e1}=A_{c1}+\frac{V_{ec}^2}{EC}+A_{ec1}^t$$

$$=A_{f1}+\frac{V_{ef}^2}{CF}+A_{ef1}^t.$$

Of these six components all except the last one in each equation are known, and the two unknown components are perpendicular

to CE and FE respectively. Thus the image of E is located at e_1. But the image of E must lie on the line $e'e'$, and therefore the original assumption that the image of B falls at b_1 is wrong.

Assume a second position b_2 for the image of B, and repeat the process. This gives a second position e_2 for the image of E. By repeated trials a series of points e_3, e_4, etc., is found. A curve drawn through e_1, e_2, e_3, etc., gives a second locus for the image of E. The intersection of this locus with $e'e'$ gives e, the true image of E. The remainder of the polygon is easily completed.

To avoid complication of the figure, only two trial positions are shown worked out. The locus $e_1e_2e_3$ is found to be a straight line. The completion of the figure is left to the reader.

87. Accelerations in Cams.—In Fig. 156 let the motion of cam 2 be completely known, and let it be required to find the accelerations in cam 3.

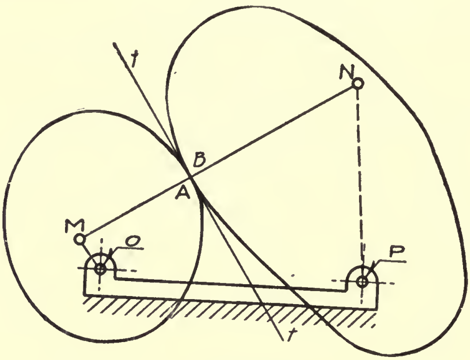


Fig. 156.

The relative motion between the cams at A and B is either,

(1) A rolling of one cam on the other;
(2) A sliding of one cam over the other;
(3) A combination of (1) and (2).

Let M and N be the centers of curvature of the parts of the two cams which are in contact. Suppose M and N were connected by a rigid link MN. Then for a very small displacement the faces of the two cams would be free to have any of the three kinds of relative motion mentioned. As the acceleration of M

is known, that of N is readily found by treating $OMNP$ as a quadric chain. When the acceleration of N has been determined that of any other point B on link *3* is found by the proportion

$$A_b : A_n = PB : PN.$$

Of course a moment later, when different points are in contact, the centers of curvature will be different, and the process must be repeated.

CHAPTER V

INERTIA FORCES OF MACHINE PARTS

88. General Statement.—The importance of the study of accelerations in mechanisms is due to the fact that this study makes possible the determination of the forces which cause the accelerations. These forces tend to produce vibrations or shaking of the machine, and often cause considerable stresses in machine elements which should be taken into account in design of these parts.

All moving parts of a machine are subject to accelerations, and the forces necessary to produce the accelerations must generally be applied by other members of the machine. Conversely the moving parts resist the accelerations, and thereby produce forces, known as "inertia forces," which are transmitted to the constraining elements.

In many machines it is desirable to eliminate or neutralize the inertia forces as far as practicable. In most mechanisms this would require the addition of moving bodies and would consequently complicate the machine. In the majority of such cases the benefits to be derived from reducing the inertia forces would not compensate for the extra cost and complexity of the machine. In such cases the only available remedy for excessive inertia forces lies in proper design, by means of which the mass and accelerations of the moving parts are kept within reasonable limits. In one important class of machines, however—multi-cylinder steam and internal combustion engines—it is often possible to minimize the inertia forces without adding greatly to the expense or complexity of the mechanism. This problem has received much attention, and is treated fully in the following chapter.

89. Acceleration Produced by a Single Force.—Consider any rigid body M, Fig. 157, which is constrained by outside mechan-

ism (not shown) to have any desired plane motion. This motion must be produced by forces F_1, F_2, F_3, etc., acting at points A, B, C, etc., where the body M is in contact with other members of the machine, together with the weight and other external forces acting on the link. These forces can be combined into a single resultant F having the line of action shown in Fig. 157. In other words the total acceleration of the mass M can be produced by a single force F whose magnitude, direction and line of action are indicated.

The motion of the body is not affected by the introduction of any system of balanced forces. It is, therefore, permissible to introduce at G, the center of gravity of M, two opposed forces

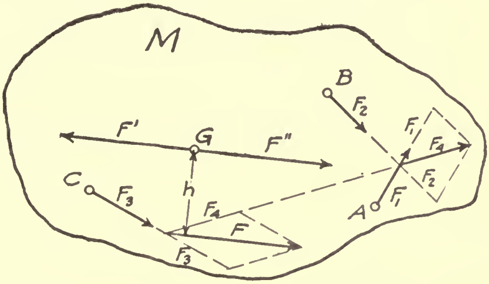


Fig. 157

F' and F'' which are each equal and parallel to F. The force F'' produces an acceleration of the center of gravity such that

$$F''=F=mA_g,$$

where m=the mass of M, and A_g is the acceleration of G. The forces F and F' form a couple of moment Fh which produces an angular acceleration of M such that

$$Fh=I\alpha=mk^2\alpha,$$

where I=moment of inertia, and k=radius of gyration of M about G as a pole, and α is the angular acceleration. It is evident that by a proper choice of F and h any desired linear and angular acceleration of M can be produced by means of the single force F.

Three special cases may arise:

(1) The forces F_1, F_2, F_3 . . . , may form a balanced system. In this case the body has momentarily no angular acceleration and no acceleration of the center of gravity. An example of this condition is a flywheel rotating at uniform speed.

(2) The resultant of the forces F_1, F_2, F_3 . . . , may be a couple. Here the center of gravity has momentarily no acceleration. The body M, however, undergoes angular acceleration.

(3) The resultant F may pass through G. In this case there is no angular acceleration.

These special cases present no difficulties. The laws derived for the general case apply without modification.

90. Kinetically Equivalent Systems.—In the case of an actual machine part the forces F_1, F_2, F_3 . . ., are unknown. The accelerations, however, may be found by the methods of Chapter IV, and the problem to be solved consists in determining the forces which cause these accelerations.

To find the resultant F completely, three things must be known:

(*a*) The magnitude of F,
(*b*) The direction of F,
(*c*) The line of action of F.

The first two of these are readily found, since

(*a*) $F = mA_g$.
(*b*) The direction of F is the same as that of A_g.

The most convenient method of finding the line of action of F is to substitute for the link M what is known as a *kinetically equivalent system*, which may be defined as a group of bodies, rigidly connected together, which will be given the same accelerations as the actual link under the action of the same forces. To meet this requirement three conditions must be fulfilled:

(*a*) The two systems must have the same mass.
(*b*) The two systems must have the same center of gravity.
(*c*) The two systems must have the same moment of inertia.

The simplest kinetically equivalent system which can be substituted for the actual link is shown in Fig. 158. It consists of

two heavy particles m_1 and m_2 connected by a weightless link. Then to satisfy the conditions of equivalence

$$m_1+m_2=m, \qquad (a)$$

$$m_1h_1=m_2h_2, \qquad (b)$$

$$m_1h_1^2+m_2h_2^2=I=mk^2. \qquad (c)$$

It will be noted that there are three equations to be satisfied and four unknowns—m_1, m_2, h_1, h_2. Therefore one of these unknowns

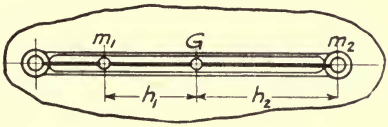


FIG. 158

can be assumed to have any convenient value, and the equations can be solved to find the other three. This principle will be used later.

Eliminating the masses m_1, m_2, and m these equations reduce to

$$h_1h_2=k^2.$$

Assuming any convenient value for h_1, h_2 can be found, thus locating the masses m_1 and m_2. In Fig. 159 let the masses m_1

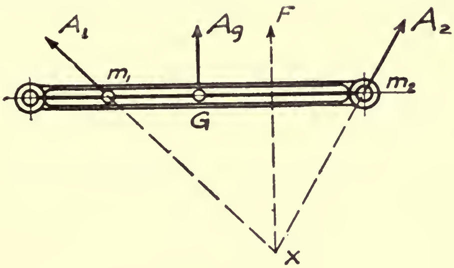


FIG. 159.

and m_2 be located as shown. Let A_1, A_2, and A_g be the accelerations of m_1, m_2, and G respectively as found from the acceleration diagram. Then the accelerations of m_1 and m_2 may be

regarded as produced by two forces $F_1 = m_1A_1$, and $F_2 = m_2A_2$ respectively, passing through m_1 and m_2 and lying in the directions of A_1 and A_2. Their resultant F must pass through the intersection X of the lines of action of these two forces. Since $F = mA_g$ the resultant is now known in magnitude, direction, and line of action.

It is customary to choose the location of m_1 at some point whose acceleration is already known, and to determine the location of m_2 by means of the relation

$$h_1h_2 = k^2.$$

For example, in studying the inertia forces of the connecting rod of a steam engine it will be found convenient to locate m_1 at the wrist pin. The line of action of F_1 is then the line of the stroke.

It should be noted that in the preceding construction it is not necessary to determine the values of the masses m_1 and m_2 or of the forces F_1 and F_2. The only requirement is to find X, the intersection of the lines of action of these forces. This point fixes the line of action of F.

91. Calculation of the Line of Action of the Resultant.—In some cases it is more convenient to calculate the distance h, Fig. 157, rather than to locate the line of action of F by means of a kinetically equivalent system.

Let ABC, Fig. 160, represent the link M and let abc be its acceleration image drawn from the pole O. ba represents the

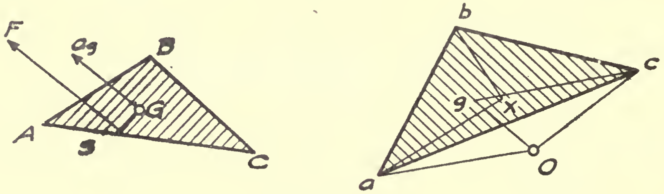


Fig. 160.

acceleration of A relative to B. This may be resolved into two components:

$$xa = AB\omega^2 \text{ parallel to } AB,$$

and

$$bx = AB\alpha \text{ at right angles to } AB,$$

where ω and α are the angular velocity and angular acceleration of link M. According to Equation (2), Art. 89

$$Fh = I\alpha,$$

or

$$mA_g h = mk^2\alpha.$$

Therefore

$$h = \frac{k^2\alpha}{A_g} = \frac{k^2 \cdot bx}{Og \cdot AB} = GS. \quad \text{(Fig. 160.)}$$

F must, of course, be parallel to Og. Therefore its line of action is fixed. To determine on which side of G the distance h must be laid off consider that the tangential acceleration of A relative to B is in the direction bx, and the angular acceleration α is therefore in the clockwise sense.

In problems where the accelerating forces are to be found only for a single position of the mechanism this method of finding the line of action is more convenient than the use of a kinetically equivalent system. Where the forces are to be found for a number of configurations of the mechanism the latter method is usually preferable.

92. Components of the Resultant Force.—The force F is the resultant of all the forces acting on the link. These are of two kinds:

(1) Pressures applied by other members of the mechanism.
(2) External forces, such as weight, pressure of steam or gas on pistons, etc.

The forces of class (2) are usually given by the conditions of the problem. They can then be subtracted vectorially from the resultant F, leaving as a remainder the resultant of the forces of class (1). The methods of splitting up this remainder into its components and thus determining the pressures at the pairing elements are illustrated in the following articles. The methods there developed will suffice to determine forces in practically any plane mechanism.

93. The Steam Engine Mechanism.[1]—Fig 161 represents the steam engine mechanism. The pressure P on the piston, link *4*,

[1] The results of a complete investigation of a standard six-cylinder gasoline motor are given in Note E.

is known. The crank, link *2*, rotates at constant angular velocity ω. The weights, centers of gravity, and moments of inertia of all links are known. It is required to find the tangential component T of the force acting at A—in other words the turning effort.

Since the crank OA rotates at uniform angular velocity the length OA can be taken to represent both the velocity and acceleration of the pin A, the velocity being revolved through 90° and the acceleration through 180°. Then OAD is the velocity polygon, OD representing the velocity of the piston, and AD the relative velocity between crank and piston. By Klein's construction OC

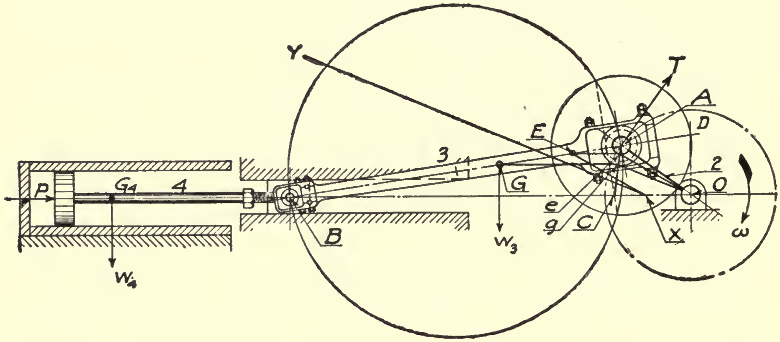


Fig. 161.

represents the acceleration of the piston, reversed in direction, and AC is the acceleration image of the rod AB.

Consider first the forces acting on the piston, link *4*. These are:

(*a*) The weight W_4, acting at G_4 the center of gravity of the piston.

(*b*) The steam pressure P acting horizontally along the center line of the rod.

(*c*) The reactions of the cylinder wall and crosshead guide. These are vertical if friction is neglected, but their point of application is not known.

(*d*) The pressure between the wrist pin and the connecting rod. The point of application of this force is at B, but its direction and magnitude are unknown.

This force may be regarded as made up of two components:

(1) A thrust in the direction AB,

(2) A force acting across the rod, and due to the weight and inertia of the rod. This force can be completely determined as shown below.

(e) The resultant of the forces (a), (b), (c) and (d) is known. This resultant acts horizontally through B and has the magnitude

$$F = m_4 A_b.$$

Where

$$m_4 = \text{mass of link } 4,$$

$$A_b = \text{acceleration of the piston} = Oc, \text{ Fig. 161.}$$

The determination of the forces on the piston now involves the following steps:

(1) Determination of the inertia forces of the rod.

(2) Combination of this inertia force with the weight of the rod.

(3) Resolution of this resultant into two parallel components acting at A and B.

(4) Combination of the known forces acting on the piston.

(5) Determination of the rod thrust and the guide reaction.

The inertia force of the rod is given by the equation

$$F = m_3 A_g,$$

where m_3 is the mass of the rod and A_g is the acceleration of G, the center of gravity of the rod. Its direction is the same as that of A_g. To determine the line of action of this force replace the rod by a kinetically equivalent system consisting of two heavy particles located at B and E, Fig. 161. According to Art. 91

$$BG \times GE = k^2,$$

where k is the radius of gyration of the rod about G. Then the inertia force may be regarded as the resultant of the inertia forces of the particles D and E. The latter act in the directions of the accelerations of these two points. Draw Ee parallel to OB. Then Oe is the acceleration of E, since AC is the image of the rod

AB. Draw EX parallel to Oe. This is the line of action of the inertia force of E. The acceleration of B is horizontal. Hence BX is the line of action of the inertia force of B. The intersection X lies on the line of action of the resultant of these forces, in other words of the inertia force of the rod. Draw Gg parallel to OB. Then Og is the acceleration of G. Through X draw XY parallel to Og. This is the line of action of the inertia force of the rod. The magnitude of this force is given by the relation

$$F = m_3 A_g.$$

Next this force must be combined with the weight of the rod. The weight W_3 acts vertically at G_3. To combine these two draw G_3S, Fig. 162, vertically to intersect XY at S. Let SY represent

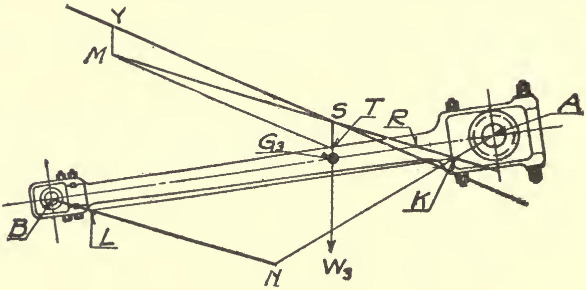


Fig. 162.

the inertia force, or the accelerating force reversed in direction. Let ST be the weight of the rod. Then SM is the resultant of these two forces.

To resolve SM into two parallel components acting at A and B draw BN equal and parallel to SM, and join AN. Prolong SM to cut AB at R and AN at K. Through K draw KL parallel to AB. Then the resultant SM can be considered as formed of two components, LB acting at B and NL acting at A. For triangles ABN and ARK are similar.

Therefore

$$\frac{LN}{LK} = \frac{LN}{BR} = \frac{KR}{AR} = \frac{BL}{AR}.$$

Or

$$LN \times AR = BL \times BR$$

Also

$$BL + LN = BN = SM.$$

Therefore LN and BR are the components of SM acting at A and B respectively.

Now returning to the piston, link *4*, the following forces are known:

(*a*) Pressure $P = Op$, Fig. 163.
(*b*) Weight $W_4 = pw$, Fig. 163.
(*c*) Component of inertia force and weight of rod acting at $B = BL$ (Fig. 162) $= wl$ (Fig. 163).

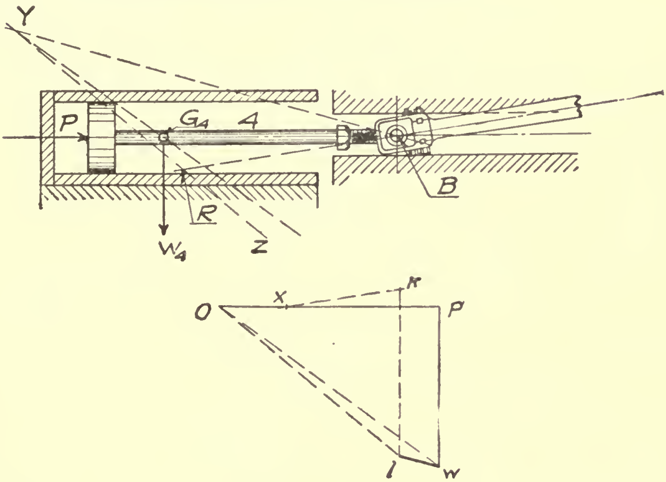


FIG. 163.

The following forces are known in direction:

(*d*) Thrust in rod AB.
(*e*) Side thrust, which is vertical neglecting friction.

The resultant of all these forces $= m_4A_4$, where m_4 is the mass of the piston and A_4 is the acceleration of the piston and is completely known. Let Ox, Fig. 163, represent this resultant. Through x draw xk parallel to AB, and through l draw lk vertical. Then xk = rod thrust and lk = side thrust.

The determination of the forces acting on the piston is now complete except for the location of the line of action of the side thrust. To find this consider that the pressure P and the weight W_4 both act through G_4. Their resultant Ow, Fig. 163, there-

fore also acts through this point. Through G_4 draw G_4Y parallel to Ow intersecting the line of action of the forces BL, Fig. 162, at Y. Through Y draw YZ parallel to Ol. YZ is then the line of action of the resultant of the pressure, the weight of the piston and that component of the inertia force and the weight of the rod which acts at B. This resultant is represented in Fig. 163 by Ol. Prolong AB to intersect YZ at R. Then R is a point on the line of action of the side thrust, since the side thrust, rod thrust and the resultant Ol must meet in a point. The forces acting on the piston are now completely known.

The force acting on the rod at A is easily found. Its two components are the rod thrust $= kx$, Fig. 163, and the second part of the inertia force and weight of the rod $= LN$, Fig. 162.

Let AK (Fig. 164) $= xk$ (Fig. 163),

and

AN (Fig. 164) $= NL$ (Fig. 162).

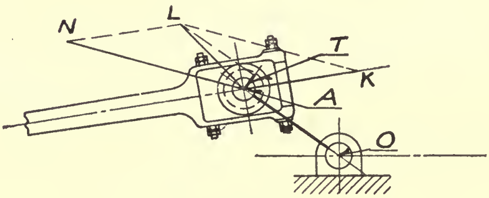


Fig. 164.

Then the resultant AL is the pin pressure at A. Resolving this into two components,

(a) AT perpendicular to the crank OA,
(b) LT parallel to the crank.

It is evident that AT represents the turning effort, and LT the thrust in the crank. If the thrust LT is combined with the weight and inertia force of the crank the resultant gives the bearing pressure, or the force which the bearing must exert in order to hold the crank in place.

Note.—All of the constructions in Figs. 162, 163, 164, could have been placed on Fig. 161. It was thought best, however, for the sake of clearness in showing the successive steps in the solution, to separate the constructions. In applying the principles set forth in the preceding paragraphs it is usually more convenient to do all the work on a single figure.

94. The Atkinson Gas Engine.—In the Atkinson gas engine, Fig. 165, the crank, link *2*, revolves at constant angular velocity ω. The pressure P on the piston, link *6*, is known, as well as the weights, centers of gravity and moments of inertia of all the links. It is required to find the turning effort exerted at the crank pin A.

The velocity polygon, Fig. 166, and the acceleration polygon,

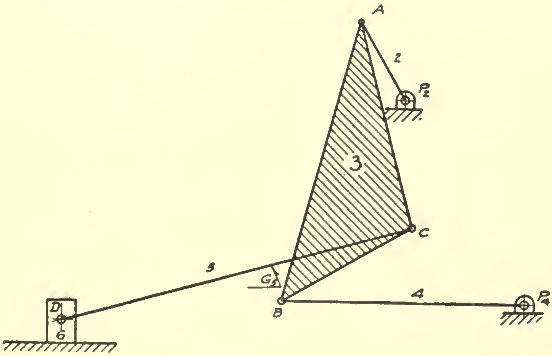


FIG. 165.

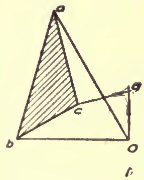


FIG. 166.

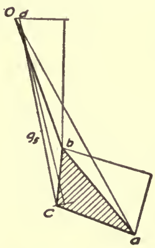


FIG. 167.

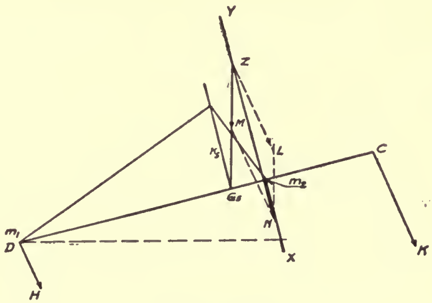


FIG. 168.

Fig. 167, are drawn by the methods of the preceding chapters. The first part of the solution follows exactly the same procedure as in the ordinary steam engine. The inertia force of the rod is found, combined with the weight, and resolved into parallel components acting at the ends of the rod. The construction is shown in Fig. 168, the method followed being exactly the same as that employed in Fig. 162. The forces on the piston are then

shown in Fig. 169 in the same way as in Fig. 163, the only difference being that since the center of gravity is taken at D the side thrust acts through this point, so that it is not necessary to determine its line of action. In Fig. 170 the pressure CL on pin C is found in the same manner as the crank pin pressure in Fig. 164.

The next step is the determination of the forces acting on links *3* and *4*. The forces acting on link *3* are as follows:

(*a*) The weight W_3.
(*b*) The pin pressure at C.
(*c*) The pin pressure at B.
(*d*) The pin pressure at A.

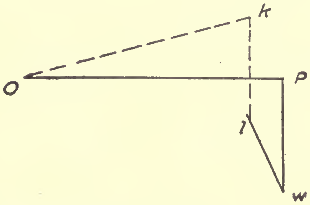


FIG. 169.

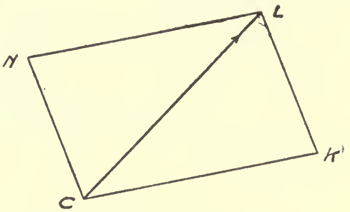


FIG. 170.

Of these (*a*) and (*b*) are completely known. The points of application of the forces (*c*) and (*d*) are at the pins B and A. The resultant

$$F_3 = m_3 A_g$$

where m_3 is the mass of the link, and A_g is the acceleration of the center of gravity G_3. F_3 is thus known in magnitude and direction, but its line of action is still to be found. The pin pressure at B can be resolved into a component in the direction P_4B and two components due to the acceleration and weight of link *4*. The latter components can be comple ely determined. Let P_4B, Fig. 171, represent link *4* and Ob its acceleration image. Substitute for the link a kinetically equivalent system consisting of two particles located at P_4 and M. Then $P_4G_4 \times G_4M = k^2{}_4$ where k_4 is the radius of gyration of link *4* about G_4. Locate on Ob the image m of the point M. Since P_4 has no acceleration the inertia force of link *4* passes through M. As usual its direction and magnitude are given by the equation

$$F_4 = m_4 A_g,$$

where A_g is the acceleration of the center of gravity G_4. A_g is given by Og_4, Fig. 171. The force F_4 is therefore completely known and is represented by MS, Fig. 171. F_4 is the resultant of the weight of the link and the pin pressures exerted at P_4 and B. Through G_4 d aw a vertical in ersecting the line MS at X. From X lay off $XT = MS$ and from T lay off $TY = W_4$. Then XY represents completely the vector difference between F_4 and W_4—in other words, the resultant of the pin pressures at P_4 and B. Resolve XY into two parallel components BZ and P_4W acting at B and P_4 respectively. BZ and PW are components

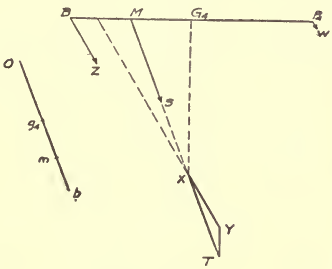


FIG. 171.

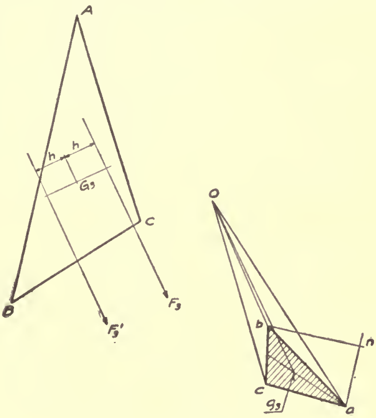


FIG. 172.

of the pin pressures at B and P_4, the other components being equal and opposite forces acting along the line P_4B. Thus another of the forces acting on link *3* has been resolved into a known component and a second force whose line of action is known.

Returning now to link *3* the next step is to locate the line of action of the inertia force F_3. In Fig. 172 le ABC represent link *3* and abc its acceleration image, taken from Fig. 167. Let G_3 be the center of gravity of link *3* and g_3 its accele ation image. Draw an parallel to AB and bn perpendicular to AB. Then $bn = AB\alpha$. It was shown in Art. 91 that $F_3h = I\alpha = m_3k_3^2\alpha$, where h is the perpendicular distance from G_3 to the line of action of F_3. Therefore

$$h = k_3^2 \frac{bn}{Og_3 \cdot AB}.$$

Since nb represents the tangential component of B's acceleration relative to A it follows that α is in a clockwise direction, and that

therefore F_3 must be located as shown in Fig. 172 and not at F'_3. If an additional force equal and opposite to F_3 were applied to the link the whole system would be in equilibrium. The forces then acting on the link would be:

(*a*) F_3 reversed in direction. (Completely known.)
(*b*) The weight W_3. (Completely known.)
(*c*) Pin pressure at C. (Completely known.)
(*d*) Pin pressure at B.
  (1) Component due to weight and inertia of link *4* $= Fb$. (Completely known.)
  (2) Thrust along P_4B. (Direction known.)
(*e*) Pin pressure at A. (Point of application known.)

All the known forces can be combined into a single resultant R as shown in Fig. 173. The link may then be considered in equilibrium under the action of three forces:

(1) The resultant R.
(2) The thrust along $P_4B = T_4$.
(3) The pin pressure at $A = Pa$.

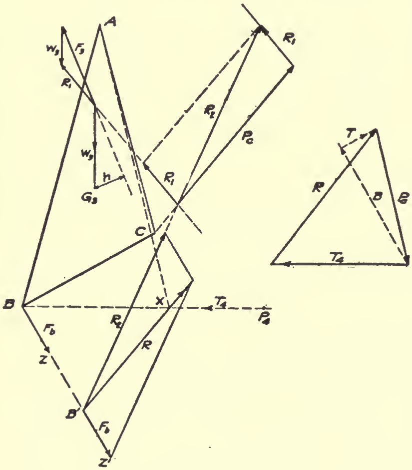


Fig. 173.

These three forces must meet in a point. Let X, Fig. 173, be the intersection of the lines of action of the first two. Then AX is the line of action of the pin pressure at A. The value of the unknown forces can now be found from the triangle of forces, Fig. 173. The pin pressure at A may be resolved into a radial component B and a tangential component T. The problem is thus completely solved.

95. The Wanzer Needle-bar Mechanism.—In the mechanism shown in Fig. 174 the disk link *2* revolves at uniform angular velocity ω. The pins A and B on the triangular link *3* move in the

slots in link *2*, and the pin C is guided in a vertical line. Given the angular velocity ω and the weight, center of gravity and moment of inertia of link *3*, it is required to find the pressures on pins A, B, and C.

The velocity polygon, Fig. 175, is constructed in the usual manner, and the acceleration polygon, Fig. 176, is found by the use of the auxiliary point M, and Coriolis' law as explained in Arts. 85 and 67.

The magnitude, direction, and line of action of the resultant accelerating force acting on link *3* is found precisely as in the Atkinson Gas Engine.

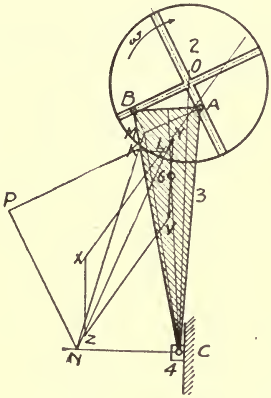


FIG. 174.

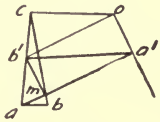


FIG. 175.

Let XY, Fig. 174, represent this force. If a force equal and opposite to this resultant is applied to the system link *3* will be in equilibrium under the action of 5 forces as follows:

(1) $F_3 = XY$. (Completely known.)
(2) $W_3 = YV$. (Completely known.)
(3) Pin pressure at A acting in line AM.
(4) Pin pressure at B acting in line BM.
(5) Pin pressure at C acting in line CN.

The first two can be combined into a single force YZ, thus reducing the number of forces to four. These forces can be combined in pairs, the resultant of one pair being equal and opposite to that of the other pair. The resultant of YZ and the pin pressure at C must pass through N, the intersection of their lines of action. Similarly the resultant of the pin pressures at A and B must pass through M.

Therefore MN is the line of action of both resultants. From N lay off $NL = YZ$ and draw LK parallel to CN. Then KL is

the pin pressure at C and KN is the resultant of this pressure and YZ. NK is then resolved into components NP and PK

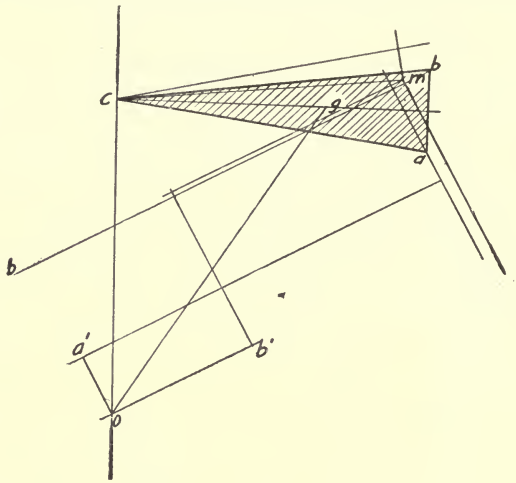


Fig. 176.

parallel to BM and AM respectively. These components represent the pin pressures at A and B.

If a known force is to be overcome at C, it is only necessary to combine this known force with the resultant YZ and proceed exactly as before.

CHAPTER VI

BALANCING OF ENGINES

96. Introductory.—One of the most important applications of the study of inertia forces is in the balancing of engines. The inertia forces in high-speed engines are of great magnitude and the balancing of them is a very important consideration. This subject has been given extensive study, the most complete treatise in the English language being by W. E. Dalby. The following discussion is based in general on the methods of Dalby.

97. Kinetic Load to an Unbalanced Mass.—Suppose a shaft S, Fig. 177, to carry a mass M whose center of gravity is at a dis-

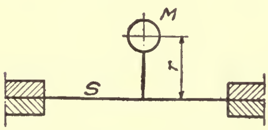


FIG. 177.

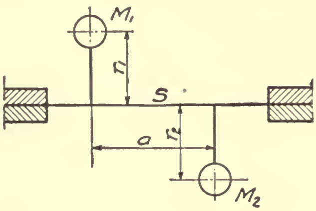


FIG. 178.

tance r from the shaft axis. If the shaft rotates with an angular speed ω, the connection between the mass and the shaft is subjected to a stress of magnitude $Mr\omega^2$. The bearings must therefore take up a *kinetic load* whose magnitude is $Mr\omega^2$ pounds. As the direction of this load is continually changing, vibrations is the frame or foundation which carries the bearings may be set up when the shaft rotates at high speed.

98. Centrifugal Couple.—If the shaft S, Fig. 178, carries two masses M_1 and M_2 in different planes of revolution, but in the same axial plane, and if further the centrifugal forces $M_1r_1\omega^2$ and

[1] The Balancing of Engines, by W. E. Dalby. Longmans, Green & Co.

$M_2r_2\omega^2$ are equal, then the shaft is subjected to the action of a *centrifugal couple.* This couple tends to turn the shaft in an axial plane and must be resisted by an equal couple applied by the bearings.

99. Masses in a Single Plane of Revolution.—If several masses, M_1, M_2, etc., Fig. 179, lie in the same transverse plane, the shaft is subjected to concurrent forces, $M_1r_1\omega^2$, $M_2r_2\omega^2$, etc., acting in the plane of revolution of the masses. The condition that the load on the shaft shall be zero is given by the equation

$$M_1r_1\omega^2+M_2r_2\omega^2 \ . \ . \ . \ M_nr_n\omega^2=0,$$

or since ω^2 is a common factor,

$$M_1r_1+M_2r_2 \ . \ . \ . \ M_nr_n=0. \qquad (1)$$

In other words, if the products M_1r_1, M_2r_2, etc., are laid off in succession as vectors, each in its proper direction, these vectors should form a closed polygon.

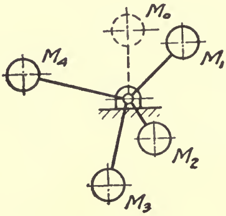


FIG. 179.

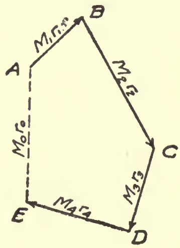


FIG. 180.

Returning again to Fig. 179, when the products M_1r_1, M_2r_2, etc., are laid off as in Fig. 180, the polygon will not in general be closed. The vector EA gives the magnitude and direction of the kinetic load M_0r_0 required to close the polygon. Hence if any convenient value of r_0 is assumed, the mass M_0 is definitely located. If desired, the gap EA may be closed by means of two or more vectors chosen at will, and the given system may thus be balanced by means of two or more masses instead of one.

Since the masses are proportional to the weights, Equation (1) may be written

$$W_1r_1+W_2r_2+ \ . \ . \ . \ +W_nr_n=0. \qquad (2)$$

100. Masses in Different Transverse Planes.—Suppose masses M_1 and M_2, Fig. 181, to be connected to the shaft S at A and B respectively. Through some point O on the shaft axis pass a transverse plane (called the *plane of reference*, or briefly, the R.P.). The mass M_1 gives rise to a kinetic load $F_1 = M_1 r_1 \omega^2$. At O introduce two equal and opposite forces F_1 parallel to this kinetic load. The load F_1 at A and the opposite force F_1 at O form a centrifugal couple whose moment is $M_1 r_1 \omega^2 a_1$. Hence the single force F_1 acting at A may be replaced by an equal and parallel force acting at O and a couple whose moment is $F_1 a_1$. Likewise the force F_2 acting at B may be replaced by an equal and parallel force F_2 acting in the R.P. and a couple whose moment is $F_2 a_2 = M_2 r_2 \omega^2 a_2$.

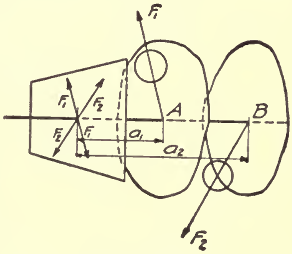


FIG. 181.

Therefore an R.P. can be chosen at will and the system of kinetic loads may be reduced to a system of concurrent forces acting in the R.P. and a system of couples acting in various axial planes. The forces have a single resultant, and the couples can be reduced to a single couple. Hence, in general, the system of kinetic loads may be reduced to a single force and a single couple. The magnitude of the couple will depend on the position chosen for the R.P.

The conditions to be satisfied in order that the shaft shall be kinetically balanced are evidently the following:

(1) The resultant centrifugal couple shall vanish; that is,

$$\omega^2(M_1 r_1 a_1 + M_2 r_2 a_2 + \ . \ . \ . \ + M_n r_n a_n) = 0;$$

(2) The resultant centrifugal force shall vanish; that is,

$$\omega^2(M_1 r_1 + M_2 r_2 + \ . \ . \ . \ + M_n r_n) = 0.$$

To balance a shaft with given revolving masses, at least two additional masses in different transverse planes are required. To determine these balancing masses proceed as follows: let the given masses be denoted by M_1, M_2, etc., Fig. 182, and the balancing masses by M_0 and M'_0. Chose the transverse planes in

which the balancing masses are to lie, and take the plane of M'_0 as the R.P. Denote by a_0, a_1, a_2, etc., the distances of the planes of M_0, M_1, M_2, etc., from the R.P. The couples $M_1r_1a_1$, $M_2r_2a_2$, etc., may now be calculated. The only unknown couple is

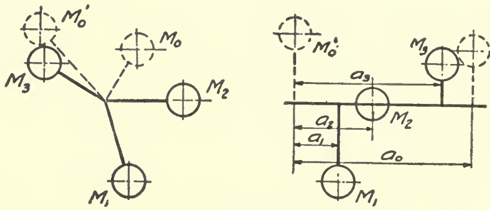


FIG. 182.

$M_0r_0a_0$, since by choosing the R.P. as the plane of the mass M'_0 the couple produced by this mass is zero. Laying off vectors representing the known couples as the sides of a polygon, Fig. 183, the closing side gives the unknown couple $M_0r_0a_0$. The arm a_0 is known, and therefore the product M_0r_0 can be found immediately. The direction of the closing couple vector gives the plane of the couple and therefore the direction of the mass M_0 from the axis.

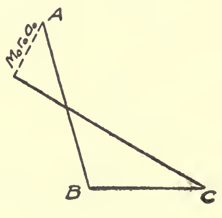


FIG. 183.

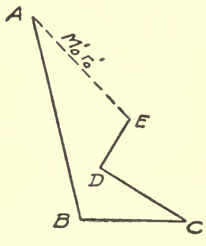


FIG. 184.

By the addition of the couple $M_0r_0a_0$ condition (1) is satisfied. It is now necessary to satisfy condition (2). Lay off the products M_1r_1, M_2r_2, etc., Fig. 184, including M_0r_0 as just found, as the sides of a polygon. The closing side gives the product $M'_0r'_0$. Choosing a convenient value for r'_0, the mass M'_0 (which must be placed in the R.P.) is readily determined. The direction of the closing side gives the direction of M'_0 from the axis.

The following are rules for drawing the sides of the polygons:

(1) The force vectors (M_1r_1, M_2r_2, etc.) are drawn from the axis outward toward the masses and parallel to the cranks.

(2) The couple vectors are likewise drawn parallel to the respective crank directions, outward for masses on one side of the R.P., and inward (toward the axis) for masses on the other side of the R.P.

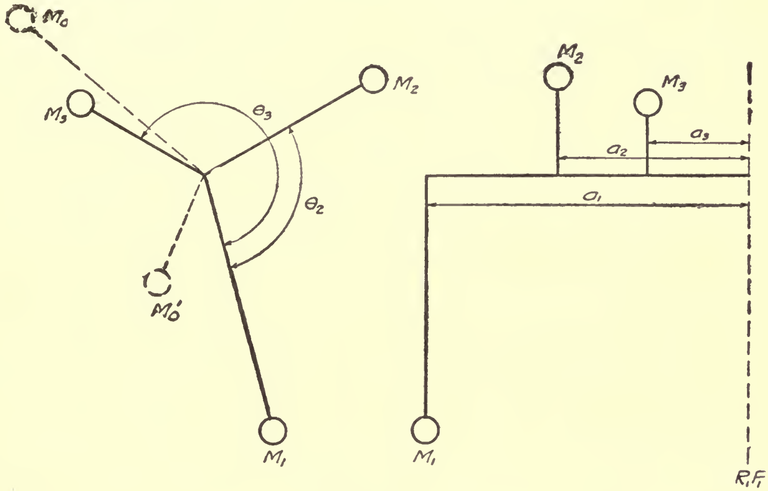


FIG. 185

Schedule I

Plane	Mass	Radius	a	θ	Mr	Mra
1	2	11	4.4	0	22	96.8
2	3	8	2.6	105°	24	62.4
3	1.5	6	1.4	225°	9	12.6
O	(2.01)	10	4.4	(216°)	(20.1)	(88.4)
O'	(1.70)	5	0	(325°)	(8.5)	0

FIG. 185.

101. Example.—Figs. 185, 186, 187 show the application of the method to an example. The masses M_1, M_2, and M_3 in the planes *1*, *2*, and *3*, Fig. 185, are to be balanced by masses in two planes. Let one of the masses be located in plane *1*. The plane of the other mass is chosen at random and is taken as the R.P. In Schedule *I* the masses, their directions, their radii, and their distances from the R.P. are entered in the proper columns. Then the products *Mr* and *Mra* are calculated and entered as shown.

The couple polygon is now drawn as shown in Fig. 186. AB is laid off in the direction of r_1 outward from the shaft, its length representing the value of the product $M_1r_1a_1 = 96.8$. Then BC and CD are laid off in order in the directions of r_2 and r_3, their lengths representing the values of the products $M_2r_2a_2 = 62.4$, and $M_3r_3a_3 = 12.6$ respectively. The closing side DA represents the couple $M_0r_0a_0$, due to the balancing mass M_0 in plane *1*. This couple is found by measurement to be 88.4; hence

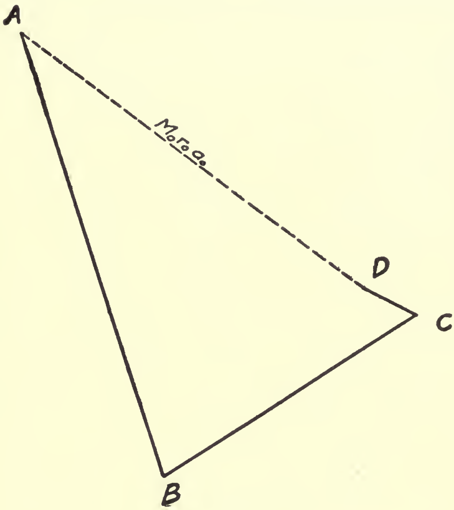


Fig. 186.

$$M_0r_0 = 88.4 = 20.1.$$

The product M_0r_0 may now be entered in its proper place in the column headed Mr. The addition of the mass M_0 at the radius r_0 in the plane *1* has made the resultant couple vanish. There are left, however, the forces in the reference plane, including the force M_0r_0, just found, and these forces will in general not be balanced. Laying off the products Mr in order, as shown in Fig. 187, the closing vector EA gives the product $M'_0r'_0$, which measures 8.5. If now the products M_0r_0 and $M'_0r'_0$ are divided by the assumed radii 10 and 5 $M_0 = 2.01$, and $M'_0 = 1.70$. The directions of the balancing masses are of course given by the directions of the closing lines DA, Fig. 186, and EA, Fig. 187.

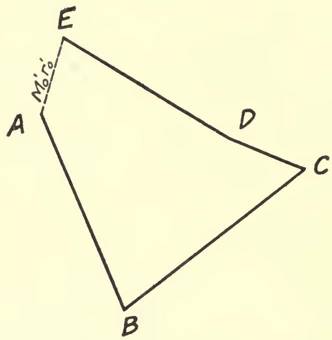


Fig. 187.

102. General Relations.—So far it has been assumed that the masses have had unequal radii. Evidently the radius of a

mass can be varied at will provided that at the same time the mass itself varies so that the product Mr remains constant. It is therefore possible to choose some convenient radius, say that of an engine crank, and all masses that have different radii can be reduced to masses having this common radius. In this case the common factor r can be dropped, the reduced masses can be used for the sides of the force polygon, and the reduced mass moments M_1a_1, M_2a_2 become the sides of the couple polygon.

In the balancing of a given system of masses, it is essential to know how many quantities may be determined beforehand, and how many must be left undetermined. For example suppose that five masses are to be placed along a shaft so as to form a balance system. If all the masses, their angles, and their distances from some R.P. are fixed at random beforehand, the system will undoubtedly be unbalanced. Some of these variables must be left for subsequent determination.

With n masses carried on a shaft, the first question to be decided is, how many variables enter into consideration. Suppose that one of the masses M_1 is fixed, that its plane of revolution is also fixed, and that its direction from the axis at some instant is assumed. The other masses may be placed in $n-1$ planes of revolution, which may be chosen arbitrarily; they may be placed in $n-1$ axial planes which make $n-1$ different angles with the plane of M_1. Evidently it is not the absolute values of the masses which count, but their ratios to each other, and therefore there will be $n-1$ mass ratios $\frac{M_1}{M_2}, \frac{M_1}{M_3}$, etc., which can be chosen arbitrarily. Hence as independent conditions there are

$n-1$ planes of revolution,
$n-1$ angles between axial planes,
$n-1$ mass ratios.

The total number of independent conditions is therefore

$$3(n-1)=3n-3.$$

To form a balanced system the force polygon and the couple polygon must both close. The closing vector in each case defines two quantities, one given by its magnitude and the other by its direction. Hence four variables can be determined by the poly-

gons. Of the $3n-3$ variables, therefore $(3n-3)-4=3n-7$ may be assumed at will, but four must be left for determination by the polygons. Thus for five masses $3\times 5-7=8$ of the variables may be assumed. For example, if the five masses are chosen, four mass ratios are fixed; if, further, three planes of revolution are assumed, one of these may be taken as the reference plane, and the distances from this plane to the other two fixes two more of the variables; finally, two angles between axial planes can be assumed. The remaining four variables—two planes of revolution and two angles—can now be found from the force and couple polygons.[1]

The fact that the conditions of balance require two closed polygons with sides parallel, but usually not in the same ratio, leads directly to some interesting and obvious conclusions. The student may verify the following statements and give reasons:

(1) Two masses cannot form a balanced system unless they are in the same plane of revolution.

(2) Three masses, to form a balanced system, must be either in the same plane of revolution or in the same axial plane.

103. Analytical Methods.—In many cases it is convenient to use analytical instead of graphical methods in determining the shaking forces and the size and position of balancing masses. For this purpose the forces are resolved into horizontal and vertical components as shown in Fig. 188. Each of the groups of forces and couples formed by this resolution must be balanced separately. The following equations then express the conditions for balance:

Horizontal forces:

$$r\omega^2(M_1 \cos \theta_1 + M_2 \cos \theta_2 + \ldots M_n \cos \theta_n) = 0.$$

Vertical forces:

$$r\omega^2(M_1 \sin \theta_1 + M_2 \sin \theta_2 + \ldots M_n \sin \theta_n) = 0.$$

[1] Freedom of choice of the $3n-7$ variables is not absolutely unrestricted. For example, if one mass is chosen greater than the sum of all the others, or if the angles are so chosen that all the masses lie on the same side of one axial plane, the system cannot be balanced. In such cases it will be found impossible to construct polygons which satisfy the assumed conditions.

Horizontal couples:

$$r\omega^2(M_1a_1 \cos \theta_1 + M_2a_2 \cos \theta_2 + \ldots M_na_n \cos \theta_n) = 0.$$

Vertical couples:

$$r\omega^2(M_1a_1 \sin \theta_1 + M_2a_2 \sin \theta_2 + \ldots M_na_n \sin \theta_n) = 0.$$

Evidently the common factor $r\omega^2$ can be omitted from all these equations. For convenience and brevity write $\cos \theta_1 = x_1$,

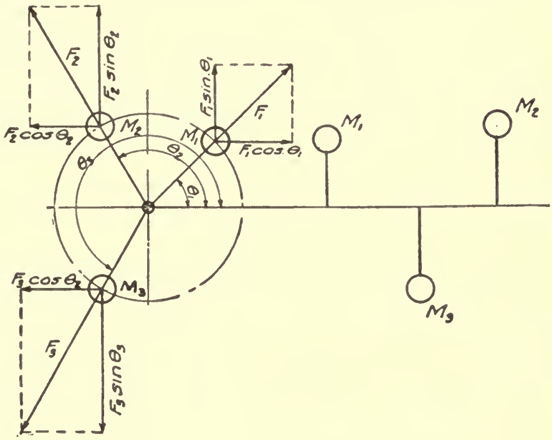


FIG. 188.

$\sin \theta_1 = y_1$, etc. Then the four equations can be conveniently written:

$$\Sigma Mx = 0, \quad \ldots \ldots \quad \text{(I)}$$

$$\Sigma My = 0, \quad \ldots \ldots \quad \text{(II)}$$

$$\Sigma Max = 0, \quad \ldots \ldots \quad \text{(III)}$$

$$\Sigma May = 0. \quad \ldots \ldots \quad \text{(IV)}$$

Equations (I) and (II) simply state that the center of gravity of the system lies in the center of the shaft. A system which satisfies Equations (I) and (II) but not Equations (III) and (IV) is in static balance but not in running balance.

104. Inertia Effects of Reciprocating Masses. Harmonic Motion.—Suppose a mass M_1, Fig. 189, to be given a reciprocating harmonic motion by means of a crank C_1. If ω is the angular speed of the crank, and θ_1 is the angle which the crank makes with the line of motion of M_1 then the radial acceleration of the crank pin is $r\omega^2$, and the horizontal component of this acceleration is $r\omega^2 \cos \theta_1$. This horizontal component is also the acceleration of M_1. The truth of this statement is almost self-evident,

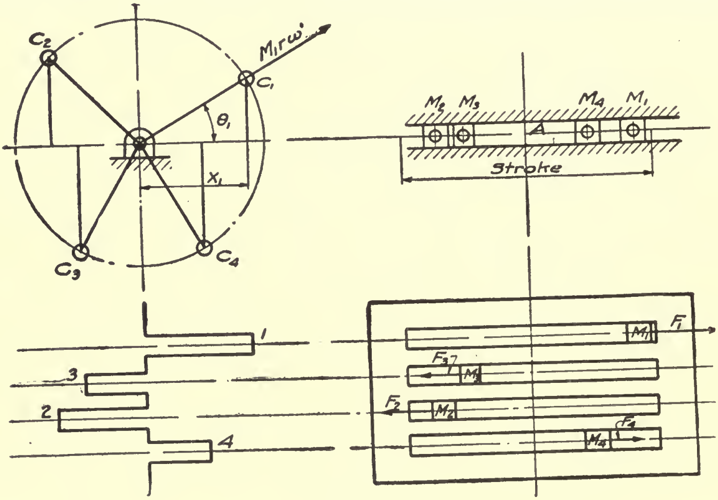


Fig. 189.

but it may be shown analytically as follows: the displacement of M_1 from the middle point of its motion is

$$x = r \cos \theta_1.$$

Hence

$$\frac{dx}{dt} = -r \sin \theta_1 \frac{d\theta_1}{dt} = -r\omega \sin \theta_1,$$

and

$$\frac{d^2x}{dt^2} = -r \cos \theta_1 \frac{d\theta_1}{dt} = -r\omega^2 \cos \theta_1.$$

The direction of the acceleration of M_1 is always toward A, the mid-point of its stroke. The accelerating force required is

$$M_1 r\omega^2 \cos \theta_1,$$

and this force must likewise be directed toward A.

If the reciprocating masses M_1, M_2, etc., are the piston and crossheads of an engine, then forces equal and opposite to the accelerating forces act on the frame or bed of the engine. Suppose that the engine has four cranks and four reciprocating masses as shown in Fig. 189. Then the engine frame will be subject to action of four inertia forces,

$$F_1 = M_1 r\omega^2 \cos\theta_1,\ F_2 = M_2 r\omega^2 \cos\theta_2,\ \text{etc.},$$

all of which will be directed from the mid plane outward. Of course all these forces lie in one plane, the plane of reciprocation.

The system of parallel forces may be treated in the same manner as the system of forces along a shaft, Art. 100. A reference plane is chosen perpendicular to the plane of the reciprocating masses. Then the force F_1 may be replaced by an equal force in the reference plane and a couple whose moment is F_1a_1, where a_1 is the distance of F_1 from the R.P. Similarly the forces F_2, F_3, etc., can be replaced by equal forces in the R.P. and suitable couples. The system is therefore reduced to a set of collinear forces in the R.P. and a system of couples in the plane of reciprocation. The forces can be reduced to a single resultant, the scalar sum

$$F_1 + F_2 + F_3 \ . \ . \ .$$

which causes a backward and forward movement of the frame as a whole. The system of couples reduces to a single couple whose moment is the scalar sum

$$F_1a_1 + F_2a_2 + F_3a_3 \ . \ . \ .$$

The effect of this couple is to rock the frame in the plane of reciprocation.

105. Balancing Conditions.—The system of forces acting on the engine frame will be balanced when the single resultant and the resultant couple are both equal to zero for every position which the masses can take. These conditions are expressed by the equations

$$F_1 + F_2 \ . \ . \ . \ F_n = 0,$$

$$F_1a_1 + F_2a_2 + \ . \ . \ . \ F_na_n = 0.$$

Or

$$r\omega^2(M_1\cos\theta_1 + M_2\cos\theta_2 + \ . \ . \ . \ M_n\cos\theta_n) = 0$$

$$r\omega^2(M_1a_1\cos\theta_1 + M_2a_2\cos\theta_2 + \ . \ . \ . \ M_na_n\cos\theta_n) = 0.$$

Evidently the factor $r\omega^2$ may be omitted from each equation. In order that the system may be balanced for every position these equations must hold when the cranks are turned through any angle ϕ. In this case the angles θ_1, θ_2, θ_3 . . . , become $\theta_1+\phi$, $\theta_2+\phi$ . . . . Substituting these values in the equations of balance we get

$$M_1 \cos(\theta+\phi)+M_2 \cos(\theta_2+\phi)+ \ . \ . \ . \ M_n \cos(\theta_n+\phi)=$$

$$M_1 \cos\theta_1 \cos\phi+M_2 \cos\theta_2 \cos\phi+ \ . \ . \ . \ M_n \cos\theta_n \cos\phi-$$

$$(M_1 \sin\theta_1 \sin\phi+M_2 \sin\theta_2 \sin\phi+ \ . \ . \ . \ M_n \sin\theta_n \sin\phi)=$$

$$\cos\phi(M_1 \cos\theta_1+M_2 \cos\theta_2+ \ . \ . \ . \ M_n \cos\theta_n)-$$

$$\sin\phi(M_1 \sin\theta_1+M_2 \sin\theta_2+ \ . \ . \ . \ M_n \sin\theta_n)=0.$$

In order that this equation may hold for every value of ϕ each of the quantities in parentheses must vanish separately. That is:

$$M_1 \cos\theta_1+M_2 \cos\theta_2+ \ . \ . \ . \ M_n \cos\theta_n=0,$$

$$M_1 \sin\theta_1+M_2 \sin\theta_2+ \ . \ . \ . \ M_n \sin\theta_n=0.$$

Or

$$\Sigma Mx=0,$$

$$\Sigma My=0.$$

Similarly from the equation for the couples

$$\Sigma Max=0,$$

$$\Sigma May=0.$$

In other words the conditions for balancing a system of reciprocating masses having harmonic motion are precisely the same as those for balancing the same masses concentrated at the cranks from which their motion is derived.

106. Engines with Finite Rods.—In the preceding discussion the masses were assumed to have harmonic motion. In an actual engine with connecting rods of finite length the motion is not strictly harmonic, but may be assumed to be to if the rod is not too short relative to the crank.

It is customary to assume part of the mass of the rod itself as concentrated at the crosshead pin and the remainder at the

crank pin. Thus if L is the length of the rod, and h the distance from the crosshead pin to the center of gravity, the fraction h/L will be considered concentrated at the crank pin and the remainder at the wrist pin.

Two approximations are thus introduced, (1) in considering the motion as harmonic, and (2) in dividing the mass of the rod between the two pins. It was shown in Art. 93 that if the rod is replaced by two equivalent masses, one of which is placed at the crosshead pin, the other cannot in general be at the crank pin. Later it will be shown how the first of these approximations may in some cases be corrected. A correction of the second is not possible.

107. Acceleration of Reciprocating Masses. Finite Rod.— The deviation of the actual inertia forces from those which would exist in the case of true harmonic motion introduces errors in the balancing which become serious when the connecting rod is short relative to the crank. Under certain conditions the masses may be so arranged that these errors practically disappear.

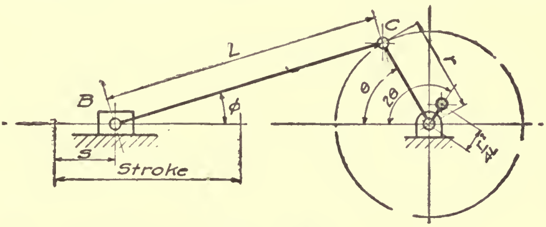


FIG. 190.

In Fig. 190 the crosshead is driven by a connecting rod of finite length L. For a given crank angle θ, measured from the inner dead center, the travel of the wrist pin B is

$$s=L+r-L\cos\phi-r\cos\theta.$$

From the geometry of the figure

$$L\sin\phi=r\sin\theta,$$

and therefore

$$L\cos\phi=\sqrt{L^2-L^2\sin^2\phi}=\sqrt{L^2-r^2\sin^2\theta}=L\sqrt{1-\frac{r^2}{L^2}\sin^2\theta}.$$

Since, however, $\frac{r}{L}$ is usually less than $\frac{1}{4}$,

$$L\sqrt{1-\frac{r^2}{L^2}\sin^2\theta}=L-\frac{r^2}{2L}\sin^2\theta \text{ approximately.}$$

Hence

$$s=r-r\cos\theta+\frac{r^2}{2L}\sin^2\theta.$$

Differentiating twice

$$\frac{ds}{dt}=r\sin\theta\frac{d\theta}{dt}+\frac{r^2}{L}\sin\theta\cos\theta\frac{d\theta}{dt}=r\omega\left(\sin\theta+\frac{r}{2L}\sin 2\theta\right),$$

and

$$\frac{d^2s}{dt^2}=r\omega\left(\cos\theta+\frac{r}{L}\cos 2\theta\right)\frac{d\theta}{dt}=r\omega^2\left(\cos\theta+\frac{r}{L}\cos 2\theta\right).$$

The inertia force due to a mass M having this acceleration is

$$Mr\omega^2\left(\cos\theta+\frac{r}{L}\cos 2\theta\right).$$

This inertia force, it will be noted, can be divided into two parts,

(1) $$Mr\omega^2\cos\theta,$$

which is precisely the inertia force of the same mass having purely harmonic motion, and

(2) $$M\frac{r^2}{L}\cos 2\theta\omega^2,$$

which is the error introduced by the finite rod. The second term may be written

$$M\left(\frac{r^2}{4L}\right)(2\omega)^2\cos 2\theta.$$

This is evidently the inertia force which would be developed by a mass M which is given harmonic motion by a crank of length $\frac{r^2}{4L}$ rotating at a speed 2ω.

The actual inertia force caused by the reciprocation of the mass M may thus be separated into a *primary* and a *secondary* part. The primary part is the projection on the line of stroke of the centrifugal force of the mass M transferred to the crank

pin, while the secondary part is the projection on the line of the stroke of the centrifugal force which would result if the mass M were transferred to the crank pin C_2 of an imaginary crank of length $\frac{r^2}{4L}$ rotating in the same plane as the main crank, but at double the speed.

108. Secondary Balance.—In order that complete secondary balance may be secured two conditions must be satisfied for every value of the angles $\theta_1, \theta_2, \ldots \theta_n$.

$$\Sigma M(2\omega)^2(r^2/4L)\cos 2\theta = 0 \quad \text{or} \quad \Sigma M \cos 2\theta = 0 \quad . \quad . \quad (a)$$

$$\Sigma aM(2\omega)^2(r^2/4L)\cos 2\theta = 0 \quad \text{or} \quad \Sigma aM \cos 2\theta = 0 \quad . \quad . \quad (b)$$

Since these equations must hold for every value of θ, that is for every possible position of the masses, it is necessary as before to satisfy four equations giving the conditions for balance as follows:

$$\Sigma M \cos 2\theta = 0 \quad \text{or} \quad \Sigma M(x^2 - y^2) = 0,$$

$$\Sigma M \sin 2\theta = 0 \quad \text{or} \quad \Sigma Mxy = 0,$$

$$\Sigma Ma \cos 2\theta = 0 \quad \text{or} \quad \Sigma Ma(x^2 - y^2) = 0,$$

$$\Sigma Ma \sin 2\theta = 0 \quad \text{or} \quad \Sigma Maxy = 0.$$

For complete balance of primary and secondary forces and couples eight equations must therefore be satisfied.

$$\left.\begin{array}{ll} \Sigma Mx = 0 \quad . \quad . \quad . \quad . \quad . \quad . & \text{(I)} \\ \Sigma My = 0 \quad . \quad . \quad . \quad . \quad . \quad . & \text{(II)} \end{array}\right\} \text{Primary forces balanced.}$$

$$\left.\begin{array}{ll} \Sigma Max = 0. \quad . \quad . \quad . \quad . \quad . & \text{(III)} \\ \Sigma May = 0. \quad . \quad . \quad . \quad . \quad . & \text{(IV)} \end{array}\right\} \text{Primary couples balanced.}$$

$$\left.\begin{array}{ll} \Sigma M(x^2 - y^2) = 0. \quad . \quad . \quad . & \text{(V)} \\ \Sigma Mxy = 0. \quad . \quad . \quad . \quad . \quad . & \text{(VI)} \end{array}\right\} \text{Secondary forces balanced.}$$

$$\left.\begin{array}{ll} \Sigma Ma(x^2 - y^2) = 0 \quad . \quad . \quad . & \text{(VII)} \\ \Sigma Maxy = 0 \quad . \quad . \quad . \quad . \quad . & \text{(VIII)} \end{array}\right\} \text{Secondary couples balanced.}$$

109. Partial Balance.—In order that an engine may be in complete primary and secondary balance, the eight equations of

the preceding article must be satisfied—that is, eight variables must be left undetermined. As the number of variables at our disposal is $3(n-1)$ it follows that no engine of less than four cranks can be completely balanced. Practically five cranks is the minimum, the solution for the four-crank engine involving an impossible arrangement from the standpoint of construction.

For the smaller numbers of cranks partial balance may be secured by satisfying part of the equations of Art. 108. These equations must always be taken in pairs. It is impossible, for example, to satisfy Equation (I) for all positions of the cranks unless Equation (II) is likewise satisfied.

The conditions for balancing the forces must always be fulfilled before attempting to balance the couples. Any solution which apparently balances the couples leaving the forces unbalanced is illusory. If the forces are not balanced the couples may disappear with respect to one particular reference plane, but not with respect to other planes. Such a solution simply shows that the unbalanced resultant force lies in the chosen reference plane. It is of course useless to attempt to get rid of the secondary forces if the much larger primary forces are left unbalanced. Therefore, in attempting to balance an engine partially it is always necessary to satisfy first Equations (I) and (II) of the preceding article. Either the primary couples or the secondary forces may then be balanced. In other words after Equations (I) and (II) are satisfied either Equations (III) and (IV), or Equations (V) and (VI) may be taken next. The secondary couples must be left to the last.

110. The Single-crank Engine.—In this case

$$n=1,$$

$$3(n-1)=0.$$

Hence none of the eight equations of balance can be satisfied. A modification of the shaking forces in such an engine can, however, be effected by *counterbalancing*. The primary force in such an engine is $Mr\omega^2 \cos\theta$. If a counterweight M_0 is placed upon the shaft at a distance from the center r_0 such that $M_0r_0=Mr$ and directly opposite the crank pin, this weight will exert a centrifugal force $Mr\omega^2$ in a direction opposite to the crank radius.

Resolving this force into horizontal and vertical components there results,

$$\text{Horizontal force} = -Mr\omega^2 \cos \theta.$$

$$\text{Vertical force} = -Mr\omega^2 \sin \theta.$$

The primary shaking forces are thus balanced and vertical shaking forces substituted.

If a smaller counterweight is used such that $M_0r_0 = kMr$ where k is less than 1 the primary horizontal force becomes

$$Mr\omega^2 \cos \theta - kMr\omega^2 \cos \theta = Mr\omega^2(1-k) \cos \theta$$

and the vertical shaking force

$$-kMr\omega^2 \sin \theta.$$

EXERCISE

1. A 10×12 inch single-cylinder engine runs at 250 r.p.m. Length of connecting rod = 36 inches. Weight of piston, crosshead, etc., including portion of connecting rod which is considered concentrated at wrist pin = 100 pounds. Determine shaking forces for each 30° of the revolution, using $k=0$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1. Draw a 3-inch circle representing the crank circle, and mark the 30° points. From each of these points draw vectors outward, representing the shaking forces reversed in direction. Join the ends of these vectors by a smooth curve. Repeat for each value of k. Scale of forces 200 pounds = 1 inch. The resulting curves show the effect of various counterweights.

111. The Two-crank Engine.—Here

$$n=2.$$

$$3(n-1)=3.$$

Since the equations of Art. 108 must be taken in pairs only (I) and (II) can be satisfied. These equations reduce to

$$M_1x_1+M_2x_2=0.$$

$$M_1y_1+M_2y_2=0.$$

These equations can be satisfied by making

$$M_1=M_2.$$

$$-x_1=x_2.$$

$$-y_1=y_2.$$

In other words the reciprocating masses should be equal and the cranks opposite to one another. This arrangement is usually undersirable on account of the uneven turning effort, and the cranks are ordinarily placed at right angles.

EXERCISE

2. A duplex engine has two cranks at right angles. The sizes, weights, speed, etc., are the same as for the single cylinder engine of Exercise 1. Construct diagram similar to that in Exercise 1, showing the combined shaking forces for the two cylinders, each crank being counterweighted as before. The center lines of the two cylinders are 4 feet apart. Taking a reference plane midway between the two cylinders construct a similar diagram for the shaking couples. Scale for couples, 1 inch = 400 foot-pounds.

112. Three-crank Engine.—In this case

$$n=3.$$

$$3(n-1)=6.$$

Hence the first six equations of Art. 108 can be satisfied. Choose the position of crank 1 as horizontal so that

$$x_1=1, \quad y_1=0,$$

and take the reference plane through crank 1 so that

$$a_1=0.$$

The six equations then reduce to

$$M_1+M_2x_2+M_3x_3=0, \quad \ldots \quad \text{(I)}$$

$$M_2y_2+M_3y_3=0, \quad \ldots \quad \text{(II)}$$

$$M_2a_2x_2+M_3a_3x_3=0, \quad \ldots \quad \text{(III)}$$

$$M_2a_2y_2+M_3a_3y_3=0, \quad \ldots \quad \text{(IV)}$$

$$M_1+M_2(x^2{}_2-y^2{}_2)+M_3(x^2{}_3-y^2{}_3)=0, \quad \ldots \quad \text{(V)}$$

$$M_2x_2y_2+M_3x_3y_3=0. \quad \ldots \quad \text{(VI)}$$

From Equations (II) and (IV) either

$$x_2=x_3, \quad \text{or} \quad y_2=y_3=0.$$

The second solution is untenable since Equation (V) cannot be satisfied in this case.

Therefore

$$x_2 = x_3.$$

$$x_2^2 + y_2^2 = x_3^2 + y_3^2 = 1.$$

$$y_2 = \pm y_3.$$

If $y_2 = y_3$ Equation (II) cannot be satisfied.

Hence

$$y_2 = -y_3.$$

Now from Equation (II),

$$M_2 = M_3.$$

The six equations are now reduced to four as follows:

$$M_1 + 2M_2x_2 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I})'$$

$$M_2x_2(a_2 + a_3) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{III})'$$

$$M_2y_2(a_2 - a_3) = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{IV})'$$

$$M_1 + 2M_2(x^2{}_2 - y^2{}_2) = 0. \quad . \quad . \quad . \quad . \quad (\mathrm{V})'$$

Since x_2 and y_2 cannot be zero it follows from (III)′ and (IV)′ that

$$a_2 = a_3 = 0.$$

In other words all the cranks must lie in the same radial plane. Subtracting (I)′ from (V)′ and substituting $y_2^2 = 1 - x_2^2$,

$$2x_2^2 - x_2 - 1 = 0.$$

Therefore

$$x_2 = 1 \text{ or } -\tfrac{1}{2}.$$

If $x_2 = 1$, $M_2 = -\frac{1}{2}M_1$, which is impossible.

Hence

$$x_2 = x_3 = -\tfrac{1}{2}$$

and

$$y_2 = -y_3 = \pm\tfrac{1}{2}\sqrt{3}.$$

Therefore the cranks are at 120°, and the masses are all equal.

Since all the cranks are in the same plane this solution is of no value. We must therefore be content with satisfying only four of the conditions of balance instead of six. For this purpose either the primary forces and primary couples may be balanced, or all the forces may be balanced without regard to the couples. In the first case the first four equations must be satisfied. These are as before:

$$M_1+M_2x_2+M_3x_3=0, \quad \ldots \quad \text{(I)}'$$

$$M_2y_2+M_3y_3=0, \quad \ldots \quad \text{(II)}'$$

$$M_2a_2x_2+M_3a_3x_3=0, \quad \ldots \quad \text{(III)}'$$

$$M_2a_2y_2+M_3a_3y_3=0. \quad \ldots \quad \text{(IV)}'$$

From (II)′ and (IV)′ either $a_2=a_3$, or $y_2=y_3=0$.

The first of these alternatives reduces to the same solution as found for the case where it was attempted to satisfy six equations.

If $\qquad y_2=y_3=0. \quad x_2=\pm 1 \quad \text{and} \quad x_3=\pm 1.$

Then

$$M_1\pm M_2\pm M_3=0,$$

$$M_2a_2+M_3a_3=0.$$

These equations admit of an infinite number of solutions, all of which have the following characteristics:

(*a*) The cranks are all in one axial plane, two being parallel and the third at 180° to these two.

(*b*) One of the masses is equal to the sum of the other two, and is attached to the crank which lies opposite the other two.

(*c*) Considering the masses concentrated at their respective crank pins, the center of gravity of the two masses on the same side of the shaft lies directly opposite the third mass.

Since cranks at 180° are undesirable from the standpoint of uniformity of turning effort this arrangement is seldom used.

It is also possible to balance the primary and secondary forces disregarding the couples. For this purpose Equations (I), (II), (V), and (VI) must be satisfied. These equations become:

$$M_1+M_2x_2+M_3x_3=0, \quad \ldots \quad \text{(I)}'$$

$$M_2y_2+M_3y_3=0, \quad \ldots \quad \text{(II)}'$$

$$M_1+M_2(x_2^2-y_2^2)+M_3(x_3^2-y_3^2)=0, \quad \ldots \quad \text{(V)}'$$

$$M_2x_2y_2+M_3x_3y_3=0. \quad \ldots \quad \text{(VI)}'$$

From (II)′ and (VI)′

$$x_2=x_3, \quad \text{or} \quad y_2=y_3=0.$$

If $y_2=y_3=0$, Equation (V)′ becomes $M_1+M_2+M_3=0$, which is impossible.

Therefore

$$x_2=x_3 \quad \text{and} \quad -y_2=y_3.$$

From (II)′

$$M_2=M_3.$$

Substituting these values in (I)′ and (V)′ and combining, there results

$$2x_2^2-x_2-1=0.$$

Hence

$$x_2=\tfrac{1}{2},$$

$$y_2=\tfrac{1}{2}\sqrt{3},$$

$$y_3=-\tfrac{1}{2}\sqrt{3},$$

$$M_1=M_2=M_3.$$

That is, the masses are all equal and the cranks are at 120°. The distances along the crank were not taken into account, and the couples are unbalanced. This arrangement is much used in practice.

113. Engines Having More than Three Cranks.—For detailed discussion of the balancing of engines with more than three cranks the reader is referred to Dalby's "Balancing of Engines," where solutions for engines of four, five, and six cylinders are given in detail. Only those types which are commonly used in automotive engineering will be discussed here.

114. The Four-cylinder Automobile-type Engine.—In this type the cranks are arranged in pairs, the outer cranks being parallel and opposite to the inner cranks. The arrangement is shown in Fig. 191. Each of the two pairs of cranks is symmetrical wi h respect to the central reference plane. Accordingly

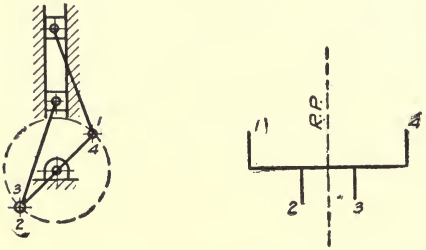


Fig. 191.

$$x_1 = x_4 = -x_2 = -x_3,$$

$$y_1 = y_4 = -y_2 = -y_3,$$

$$a_1 = -a_4,$$

$$a_2 = -a_3.$$

Substituting these values in the eight equations of Art. 108, all the equations are satisfied except (V) and (VI), which become,

$$\Sigma M(x^2 - y^2) = 4M_1(x^2{}_1 - y_1{}^2) \neq 0,$$

$$\Sigma Mxy = 4M_1 x_1 y_1 \neq 0.$$

That is to say that the primary forces and couples are balanced, but the secondary forces are not. The secondary couples are balanced with respect to the chosen reference plane, but not with respect to any other plane.

115. The Six-cylinder Automobile-type Engine. — In this engine the cranks are arranged in three pairs symmetrically disposed with respect to the central reference plane. The cranks form angles of 120° with each other. The scheme is illustrated in Fig. 192. Since all the masses are equal, and since corresponding cranks are at equal distances on opposite sides of the reference plane all eight of

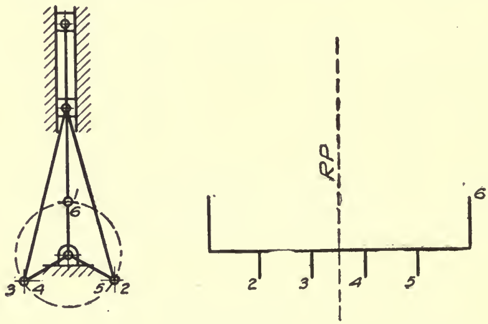


Fig. 192.

the equations of Art. 108 are satisfied. This is readily verified by substituting in these equations the values of the x's, y's and a's. For convenience in making this substitution it is best to assume one of the pairs of cranks in either a horizontal or a vertical position. Then

$$x_1 = x_6 = 1. \qquad x_2 = x_5 = -\tfrac{1}{2}. \qquad x_3 = x_4 = -\tfrac{1}{2}.$$

$$y_1 = y_6 = 0. \qquad y_2 = y_5 = \tfrac{1}{2}\sqrt{3}. \qquad y_3 = y_4 = -\tfrac{1}{2}\sqrt{3}.$$

$$a_1 = -a_6. \qquad a_2 = -a_5. \qquad a_3 = -a_4.$$

With these values all the equations are satisfied, and the engine is therefore completely balanced.

116. The Eight-cylinder Automobile-type Engine.—This engine is usually built in the form of two units each composed of four cylinders arranged as described in Art. 114. The axes of the cylinders in one group are inclined at some angle ϕ to those of the other group. In each of these units the primary forces and couples are balanced, but the secondary forces are not. Since the two sets of unbalanced secondary forces are inclined to each other at an angle ϕ, they cannot neutralize each other, and the engine is therefore not balanced as far as the secondary forces are concerned.[1]

117. The Twelve-cylinder Automobile-type Engine.—This engine is usually built up of two groups of six cylinders, each group being arranged as described in Art. 115. Since each of the groups is completely balanced in itself, the whole engine is in perfect balance with regard to both primary and secondary forces and couples.

118. The Radial Engine.—The radial or star engine is a multiple-cylinder engine having a single crank. The cylinders are all in the same radial plane, and all the connecting rods are attached by special devices to a common crank pin. The size of all the cylinders and the weights of all the reciprocating parts are equal. The general arrangement is shown in Fig. 193.

[1] It can be readily shown that if the angle ϕ is a right angle, the resultant shaking forces are in a plane at right angles to the plane bisecting the angle ϕ. The value of the unbalanced force is the sum of the components of the two individual sets of secondary forces in a direction normal to the bisecting plane.

If the engine has n cylinders, the angle between the lines of stroke of two adjacent cylinders is $2\pi/n=\phi$. Then when the common crank makes an angle α with the line of stroke of cylinder n, it makes an angle

$\alpha-\phi$ with the line of stroke of cylinder number 1.

$\alpha-2\phi$ with the line of stroke of cylinder number 2.

$\alpha-3\phi$ with the line of stroke of cylinder number 3.

. . . .

$\alpha-k\phi$ with the line of stroke of cylinder number k.

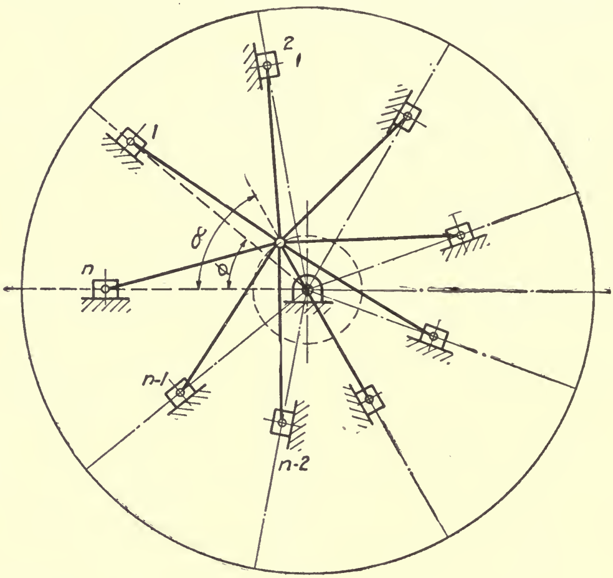


Fig. 193.

The inertia forces of the different sets of reciprocating parts then become

$Mr\omega^2$ $(\cos\alpha+r/L\cos 2\alpha)$........................for cylinder n,

$Mr\omega^2[\cos(\alpha-\phi)+r/L\cos 2(\alpha-\phi)]$................for cylinder 1,

$Mr\omega^2[\cos(\alpha-2\phi)+r/L\cos 2(\alpha-2\phi)]$..............for cylinder 2,

$Mr\omega^2[\cos(\alpha-k\phi)+r/L\cos 2(\alpha-k\phi)]$..............for cylinder k.

Taking cylinder n as horizontal, the horizontal and vertical components of the inertia forces are found to be as follows:

Horizontal forces:

$Mr\omega^2(\cos\alpha + r/L\cos 2\alpha)$ .... for cylinder n,

$Mr\omega^2[\cos(\alpha-\phi) + r/L\cos 2(\alpha-\phi)]\cos\phi$ .... for cylinder 1,

$Mr\omega^2[\cos(\alpha-2\phi) + r/L\cos 2(\alpha-2\phi)]\cos 2\phi$ .... for cylinder 2,

$Mr\omega^2[\cos(\alpha-k\phi) + r/L\cos 2(\alpha-k\phi)]\cos k\phi]$ .... for cylinder k.

Vertical forces:

$Mr\omega^2(0)$ .... for cylinder n,

$Mr\omega^2[\cos(\alpha-\phi) + r/L\cos 2(\alpha-\phi)]\sin\phi$ .... for cylinder 1,

$Mr\omega^2[(\cos(\alpha-2\phi) + r/L\cos 2(\alpha-2\phi)]\sin 2\phi$ .... for cylinder 2,

$Mr\omega^2[\cos(\alpha-k\phi) + r/L\cos 2(\alpha-k\phi)]\sin k\phi$ .... for cylinder k.

The first term in each of these expressions represents the primary force, and the second term gives the secondary force.

Let X represent the sum of the horizontal components and Y the sum of the vertical components of the primary forces.

Then

$$X = Mr\omega^2[\cos\alpha + \cos(\alpha-\phi)\cos\phi + \cos(\alpha-2\phi)\cos 2\phi\ \ldots\ ]$$

$$= Mr\omega^2\cos\alpha[1+\cos^2\phi+\cos^2 2\phi\ \ldots\ \cos^2 k\phi\ \ldots]$$

$$+ Mr\omega^2\sin\alpha[0+\sin\phi\cos\phi + \sin 2\phi\cos 2\phi \ldots \sin k\phi\cos k\phi \ldots].$$

It is readily proved that the second series reduces to zero, and the first to $n/2$.[1]

Therefore

$$X = \tfrac{1}{2}nMr\omega^2\cos\alpha.$$

$$Y = Mr\omega^2[\cos\alpha\sin 0 + \cos(\alpha-\phi)\sin\phi + \cos(\alpha-2\phi)\sin 2\phi\ \ldots\ ]$$

$$= Mr\omega^2\cos\alpha[\sin 0\cos 0 + \sin\phi\cos\phi + \sin 2\phi\cos 2\phi\ \ldots\ ]$$

$$+ Mr\omega^2\sin\alpha[\sin^2 0 + \sin^2\phi + \sin^2 2\phi\ \ldots\ ] = 0 + \tfrac{1}{2}nMr\omega^2\sin\alpha.$$

Combining X and Y, the resultant is found to be $\frac{1}{2}nMr\omega^2$, and the direction of this resultant is along the crank radius. The shak-

[1] See any standard work on trigonometry.

ing force due to the reciprocating masses can therefore be completely neutralized by a counterweight of mass $\frac{1}{2}nM$ placed directly opposite the crank and at a distance from the center equal to the crank radius.

Let X' and Y' represent the sums of the horizontal and vertical components of the secondary forces. Then

$$X'=\frac{Mr^2\omega^2}{L}[\cos 2\alpha+\cos 2(\alpha-\phi)\cos\phi+\cos 2(\alpha-2\phi)\cos 2\phi\ \ldots\ ]$$

$$=\left(\frac{Mr^2\omega^2}{L}\right)\cos 2\alpha\,[\cos 2\phi\cos\phi+\cos 4\phi\cos 2\phi\ \ldots\ \cos 2k\phi\cos k\phi]$$

$$+\left(\frac{Mr^2\omega^2}{L}\right)\sin 2\alpha[\sin 2\phi\sin\phi+\sin 4\phi\sin 2\phi\ \ldots\ \sin 2k\phi\sin k\phi].$$

Both these series vanish, and X' is therefore zero. In a precisely similar way it may be shown that the vertical force Y' vanishes.

This type of engine is therefore in complete secondary balance, and can be put in primary balance by means of a single counterweight. As all the cylinders are in the same plane, there are of course no couples to be taken care of.

It may be noted that this solution holds only when there are more than three cylinders. With two cylinders the primary forces are unbalanced. With three cylinders the secondary forces do not vanish.

119. The Opposed Engine.—In this type of engine two cranks are placed at an angle of 180°, and driven by two similar cylinders which are on opposite sides of the shaft, as shown in Fig. 194. The

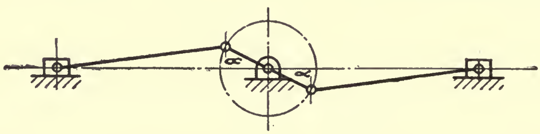


FIG. 194.

center lines of the opposed cylinders are in the same straight line.

From the symmetry of the arrangement it is evident that the motions of the two pistons are exactly equal and opposite. There-

fore the accelerations are equal, and the inertia forces exactly balance each other. Since the cylinders are in the same straight line there are no couples to be taken into account. The engine is therefore completely balanced.

The engine may have two or more pairs of opposed cylinders, each of which is completely balanced, and therefore there can be no shaking forces. It can be shown that the inertia forces of the connecting rods are also equal and opposite. Since these forces are not exactly in the same line they have a slight effect on the turning effort.

120. The Rotary Engine.—This engine is an inversion of the ordinary slider-crank mechanism, the crank becoming the stationary member, and the cylinder and connecting rod making complete revolutions about fixed centers. Usually rotary engines are made with seven or nine cylinders, all of which revolve in the same plane. By special devices all the connecting rods are attached to a common pin. The general arrangement is shown in Fig. 195.

If all the cylinders are alike, and if all the angles between adjacent cylinders are equal, evidently the revolving cylinders form a balanced system. Since all the cylinders are in the same plane there are no inertia couples. Therefore only the inertia forces of the pistons and connecting rods require investigation.

Let the number of cylinders be n.

Let $\phi = 2\pi/n$ be the angle between the center lines of adjacent cylinders, and let α be the angle between the center line of the nth cylinder and the center line of the stationary crank.

Then the angles between the axes of cylinders 1, 2, . . . k and the center line of the crank are $\alpha + \phi$, $\alpha + 2\phi$, . . . $\alpha + k\phi$, as shown in Fig. 195.

Let β be the angle between the connecting rod and the line of stroke for the nth cylinder;

r, the length of the stationary crank;

L, the length of the connecting rods;

s, the distance from the center of rotation to the center of the wrist pin of the nth cylinder;

u, the velocity of sliding of the piston in the cylinder;

and

ω, the angular velocity of rotation of the cylinders.

Then

$$s = r\cos\alpha + L\cos\beta = r\cos\alpha + L\sqrt{1 - r^2/L^2\sin^2\alpha}$$

$$= r\cos\alpha + L - r^2/2L\sin^2\alpha \text{ approximately.}$$

$$u = -ds/dt = r\omega\sin\alpha + r^2/2L\omega\sin 2\alpha.$$

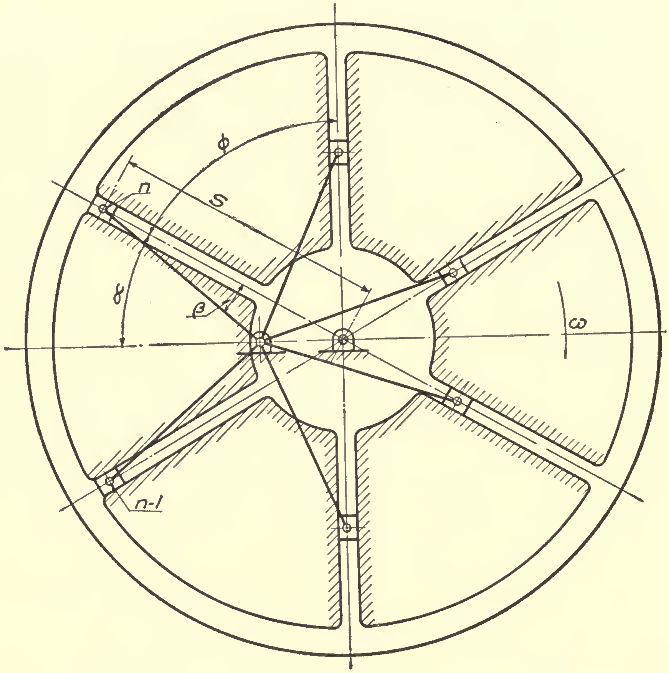


Fig. 195.

The acceleration of the wrist pin has three components, namely:

(1) Acceleration of the point on the cylinder coinciding with the center of the wrist pin. This is directed along the line of stroke and its value is $s\omega^2$.

(2) Acceleration due to sliding of piston in cylinder $= \dfrac{du}{dt}$ directed along the line of stroke.

(3) $2u\omega$ at right angles to the line of stroke.

Substituting the values of s and u these accelerations become:

(1) $s\omega^2 = r\omega^2 \cos\alpha + L\omega^2 - r^2/2L\omega^2 \sin^2\alpha.$
(2) $du/dt = -(r\omega^2 \cos\alpha + r^2/L\omega^2 \cos 2\alpha).$
(3) $2u\omega = 2r\omega^2 \sin\alpha + r^2/L\omega^2 \sin 2\alpha.$

These accelerations are shown in Fig. 196. The inertia forces are found by multiplying the accelerations by M, the mass of the piston. Evidently the common factors M and ω^2 may be omitted in discussing the balance of the engine.

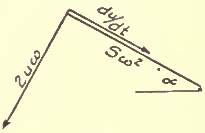


FIG. 196.

Omitting the common factors and resolving the forces into horizontal and vertical components there results:

Horizontal component.

$$X_n = s\cos\alpha + \frac{du}{dt}\frac{1}{\omega^2}\cos\alpha - \frac{2u\omega\sin\alpha}{\omega^2}$$

$$= r\cos^2\alpha + L\cos\alpha - \frac{r^2}{2L}\sin^2\alpha\cos\alpha + r\cos^2\alpha$$

$$+ \frac{r^2}{L}\cos 2\alpha\cos\alpha - 2r\sin^2\alpha - \frac{r^2}{L}\sin 2\alpha\sin\alpha$$

$$= 2r(\cos^2\alpha - \sin^2\alpha) + L\cos\alpha - \frac{r^2}{2L}\sin^2\alpha\cos\alpha$$

$$+ \frac{r^2}{L}(\cos^2\alpha - \sin^2\alpha)\cos\alpha - \frac{2r^2}{L}\sin^2\alpha\cos\alpha$$

$$= 2r\cos 2\alpha + L\cos\alpha + \frac{r^2}{L}\cos^3\alpha - \frac{7r^2}{2L}\sin^2\alpha\cos\alpha.$$

Vertical component.

$$Y_n = s\sin\alpha + \frac{1}{\omega^2}\frac{du}{dt}\sin\alpha + \frac{1}{\omega^2}2u\omega\cos\alpha$$

$$= r\cos\alpha\sin\alpha + L\sin\alpha - \frac{r^2}{2L}\sin^3\alpha + r\sin\alpha\cos\alpha$$

$$+ \frac{r^2}{L}\cos 2\alpha\sin\alpha + 2r\sin\alpha\cos\alpha + \frac{r^2}{L}\sin 2\alpha\sin\alpha$$

$$=4r\sin\alpha\cos\alpha+L\sin\alpha-\frac{r^2}{2L}\sin^3\alpha$$

$$+\frac{r^2}{L}\cos^2\alpha\sin\alpha-\frac{r^2}{L}\sin^3\alpha+\frac{r^2}{L}2\sin^2\alpha\cos\alpha$$

$$=2r\sin 2\alpha+L\sin\alpha-\frac{3r^2}{2L}\sin^3\alpha+\frac{r^2}{L}\cos^2\alpha\sin\alpha+\frac{2r^2}{L}\sin^2\alpha\cos\alpha.$$

For the other cylinders substitute for α the values $\alpha+\phi$, $\alpha+2\phi$, $\alpha+k\phi$.

Summing up the horizontal forces for all the cylinders,

$$\Sigma X=2r\Sigma\cos 2(\alpha+k\phi)+L\Sigma\cos(\alpha+k\phi)$$

$$-\frac{7r^2}{2L}\Sigma\sin^2(\alpha+k\phi)\cos(\alpha+k\phi)+\frac{r^2}{L}\Sigma\cos^3(\alpha+k\phi).$$

Each of these summations is separately zero if n is greater than 3.[1] Therefore all the horizontal forces vanish.

Similarly for the vertical forces

$$\Sigma Y=2r\Sigma\sin 2(\alpha+k\phi)+L\Sigma\sin(\alpha+k\phi)-\frac{3r^2}{2L}\Sigma\sin^3(\alpha+k\phi)$$

$$+\frac{r^2}{L}\Sigma\sin(\alpha+k\phi)\cos^2(\alpha+k\phi)+\frac{2r^2}{L}\Sigma\sin^2(\alpha+k\phi)(\cos(\alpha+k\phi).$$

Again each of these summations is equal to zero, and the vertical forces therefore also vanish.

It can also be shown that the inertia forces of the connecting rods form a balanced system. The engine is therefore completely balanced.

NOTE.—An investigation by the authors shows that the inertia forces exert no influence on the turning effort. The sum of the moments of these forces about the center O vanishes for all positions of the engine.

121. The Offset Engine.—In this type of engine the line of stroke of the wrist pin does not pass through the center of the main bearing. These engines are used in many makes of automobiles, and are built with four, six, eight, or twelve cylinders.

[1] For the case where $n=2$, the term $\Sigma\cos 2(\alpha+k\phi)$ does not vanish, and for the case where $n=3$, the terms $\Sigma\sin^2(\alpha+k\phi)\cos(\alpha+k\phi)$ and $\Sigma\cos^3(\alpha+k\phi)$ do not vanish.

The arrangements of cylinders are the same as those described in Arts. 114, 115, 116, and 117.

This engine gives a quick-return motion, the working stroke taking place while the crank revolves through the angle *ACB*,

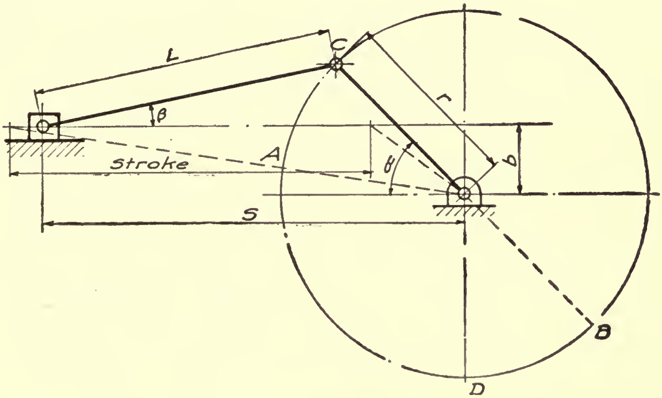


FIG. 197.

Fig. 197, and the compression and exhaust strokes lasting through the arc *BDA*. In Fig. 198 are shown the variation of velocity and acceleration for an engine having a stroke of 6 inches, a connecting rod of 15 inches and an offset of 1 inch.

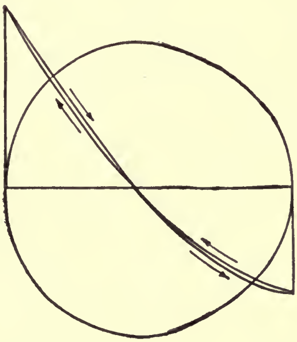


FIG. 198.

Let L = length of connecting rod;

r = crank radius;

b = offset;

S = horizontal distance from crank center to wrist pin.

Then from Fig. 197,

$$S = r\cos\alpha + L\cos\beta,$$

$$L\sin\beta = r\sin\alpha - b,$$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1 - \frac{(r\sin\alpha - b)^2}{L^2}}.$$

Expanding by the binomial theorem,

$$\cos\beta = 1 - \frac{(r\sin\alpha - b)^2}{2L^2} - \frac{(r\sin\alpha - b)^4}{8L^4} \cdot \cdot \cdot \cdot$$

$$S = r\cos\alpha + L - \frac{(r\sin\alpha - b)^2}{2L} - \frac{(r\sin\alpha - b)^4}{8L^3} \cdot \cdot \cdot \cdot$$

Differentiating twice

$$d^2S/dt^2 = \omega^2\left[-r\cos\alpha - \frac{r^2}{L}\cos 2\alpha - \frac{br}{L}\sin\alpha \right.$$
$$\left. - \frac{3(r\sin\alpha - b)^2\, r\cos^2\alpha - (r\sin\alpha - b)^3\, r\sin\alpha}{2L^3} \cdot \cdot \cdot \cdot \right].$$

The first term of this series may be called the primary acceleration, the second and third terms the secondary acceleration, and the remaining terms may be called the tertiary and higher accelerations. It is evident that the primary and secondary inertia forces and couples can be balanced under exactly the same conditions as those of ordinary engines. In other words, if the eight equations of Art. 108 are satisfied the engine is in complete primary and secondary balance.

122. The tertiary acceleration was investigated for an engine of the dimensions given above. The maximum value was found to occur when α was nearly 270° and to be about one-tenth as great as the secondary acceleration. For most purposes, this effect is too small to be considered, and the engine may therefore be regarded as balanced if the eight equations of Art. 108 are satisfied.

CHAPTER VII

GOVERNORS

123. Purpose.—The purpose of the governor is to control the supply of steam furnished to an engine, so that the machine may run at approximately constant speed under all loads. In the ideal case the supply of steam is so regulated that the speed is the same at all loads. Practically such operation is impossible, and all governors are so designed that under heavy loads the speed is less than under light loads.[1]

124. Classification.—For the purposes of analysis it is convenient to divide governors into two classes:

(1) *Flyball governors.*

(2) *Shaft governors.*

A governor of the first class has two or more weights W, Fig. 199, carried by arms A, which in turn are pivoted on a revolving shaft S. The shaft is driven at a speed proportional to the speed of the crank. As the weights revolve with the shaft they tend to move outward under the action of centrifugal force. If this force is great enough the weights W will assume a new position, and by means of the links B will move the slide w along the shaft S. The supply of steam is controlled by the position of w, decreasing as w rises.

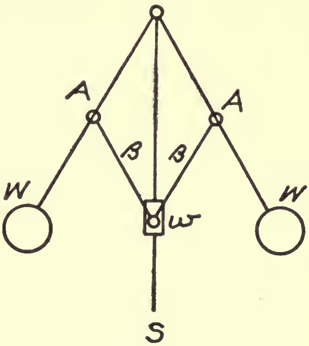


Fig. 199.

[1] Stodola (The Siemens Governing Principle, Zeitschrift des Vereines deutscher Ingenieure, 1899) has shown that under certain conditions a governor may be so designed that the engine will run at higher speed under heavy loads than under light loads. In practice no such governors are built.

A governor of the second class has a weight w, Fig. 200, placed inside the flywheel and revolving with it. The mass W is usually pivoted to one of the arms so that it can assume various positions with respect to the flywheel. As the crank revolves the mass w tends to move outward under the action of centrifugal force. This motion is opposed by the tension of the spring S. If the speed becomes sufficiently high w will move to a new position. By suitable linkage (not shown) the eccentric is shifted relative to the shaft, thus regulating the supply of steam. In addition to the centrifugal force there is another force tending to move the weight W. Suppose the flywheel and governor to be rotating together about the shaft center P with angular velocity ω. Now if the load is lightened the engine tends to speed up and the flywheel receives an angular acceleration α. Due to its inertia the governor weight resists this acceleration and thus tends to rotate about the pivot O in a sense opposite to that of α. This action is known as the *inertia* effect. Governors in which the inertia is relied upon to furnish the greater part of the regulating force are called *inertia* governors. Those where the centrifugal force predominates are called *centrifugal* governors. As will be shown later, it is possible to construct governors operating purely by centrifugal force, but it is not possible to build a governor which works by inertia only.

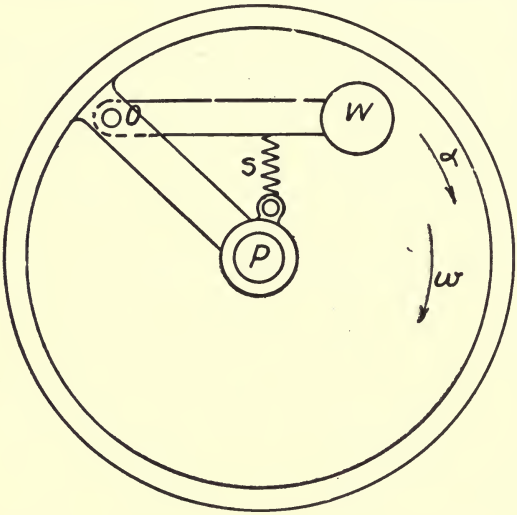


Fig. 200.

125. Force Reduction.—Suppose any body M, Fig. 201, to have constrained motion and to be acted upon by a force F applied

at the point A. Let the velocity of the point A be V_a, and let θ denote the angle between the force vector F and the velocity vector V_a. Then in the interval of time dt the force performs work $FV_a \cos \theta dt$. Let a second point B of the body have a velocity V_b. Then if a force F' is applied at B, in the interval of time dt it will do work $F'V_b \cos \phi dt$, where ϕ is the angle between the vectors V_b and F'.

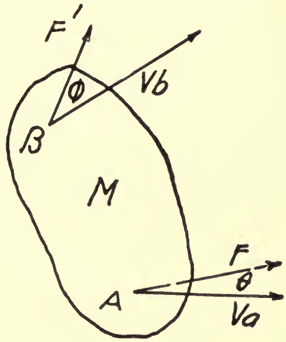


FIG. 201.

If

$$FV_a \cos \theta dt = F'V_b \cos \phi dt, \quad . \quad . \quad (1)$$

then the force F' would produce exactly the same effect on the motion of the body as the force F. Since the velocities V_a and V_b are proportional to the distances from the instantaneous center of the link to the points A and B, Equation (1) may be written

$$FR_a \cos \theta = F'R_b \cos \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where R_a and R_b are the instantaneous radii of the points A and B. The force F acting at A may therefore be replaced by the force F' acting at B without in any way altering the motion of the body.

In particular if the point B is the instantaneous center of relative motion between the link M and some other link N, then the force F acting at A on link N may be replaced by the force F' acting on link N at B, the instantaneous center of relative motion between the links M and N.

In this way all the forces acting on the various links of a mechanism may be replaced by forces acting on a single chosen link. These new forces may finally, if desired, be replaced by still other forces acting at any chosen point of the given link, and all the forces acting on the mechanism can thus be combined into a single resultant acting at any desired point. The process of replacing a given force by another acting at a different point is called *reducing* the given force to the given point. The sole condition to be observed in making such a reduction is that *for any given movement of the mechanism the new force must do the same amount of*

work as the one replaced. It should be clearly understood that the equivalence of the two forces extends only to their effect on the motion of the mechanism, and that this method is of no value in computing stresses in machine parts or shaking forces in the mechanism.

126. Mass Reduction.—Referring again to Fig. 201, suppose a mass m to be concentrated at A. This mass will have kinetic energy $\frac{1}{2}mv_a^2$. Suppose a second mass m' concentrated at B. Its kinetic energy will be $\frac{1}{2}m'v_b^2$. Then the mass m concentrated at A may be replaced by the mass m' concentrated at B provided that:

$$\tfrac{1}{2}mV_a^2=\tfrac{1}{2}m'V_b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Or since $V_a/V_b=R_a/R_b$ Equation (3) may be written

$$mR_a^2=m'R_b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus the entire mass of the link M may be *reduced* to a single mass concentrated at one point B. The magnitude of this mass is found as follows:

Let Δm be the mass at any point, and let R be the distance of this point from the instantaneous center of rotation of the link M. Then, if m_0 is the reduced mass of the entire link,

$$m_0R_b^2=\Sigma R^2\Delta m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The right-hand member of Equation (5) is the moment of inertia of the link M about its instantaneous center. Therefore Equation (5) may be written:

$$m_0R_b^2=m(R_0^2+k^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where m is the mass of link M, R_0 is the distance from the center of gravity of the link to the instantaneous center, and k is the principal radius of gyration of the link.

Thus the entire link M may be replaced by a single mass m_0 concentrated at any given point B. In particular, if the point B be chosen as the instantaneous center of relative motion between the links M and N, the link M may be replaced by the mass m_0 concentrated at the point B of the link N. In this way all the masses of all the links of a mechanism may be reduced to a system of masses attached to a single link, and finally, all these new masses may be reduced to a single mass concentrated at a single

point. If all the forces acting on the mechanism have previously been reduced to the same point, the study of the motions in the mechanism is reduced to the problem of the effect of a single force acting on a single concentrated mass.

For example, in the steam engine, all the forces and masses might be reduced to the crank pin, and the accelerations throughout the mechanism could be determined from the consideration of the action of the single resultant force on the single reduced mass. The principles of force and mass reduction will be much used in the study of the behavior of governors.[1]

127. Moment of the Centrifugal Force.—Let the mass M, Fig. 202, revolve about the axis OY with angular velocity ω, and also be free to turn about some point P. It is required to find the moment about the point P due to the centrifugal force of the mass M. Consider any elementary mass dM whose distance from OY is x, and whose distance from OPX is y. Then the centrifugal force of this elementary mass is

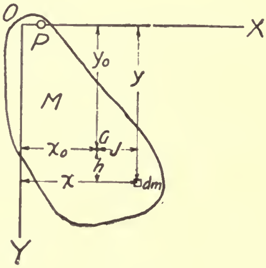


FIG. 202.

$$dC = x\omega^2 dM, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the moment of the centrigugal force about the point P is:

$$dT = x\omega^2 y dM. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let x_0, y_0 be the coordinates of the center of gravity G, and let

$$x = x_0 + j, \quad y = y_0 + h.$$

Then

$$dT = (x_0 + j)(y_0 + h)\omega^2 dM, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$T = \omega^2 \int xy\, dM = \omega^2 (x_0 y_0 M + x_0 \int h\, dM + y_0 \int j\, dM + \int jh\, dM). \quad (4)$$

[1] For formal proof of the principles of force and mass reduction see any standard work on mechanics.

But since G is the center of gravity,

$$\int h\,dM=\int j\,dM=0. \qquad (5)$$

Therefore

$$T=\omega^2 x_0 y_0 M+\omega^2\int jh\,dM.^2 \qquad (6)$$

The term $x_0y_0M\omega^2$ is the moment of the centrifugal force, assuming the entire mass concentrated at the center of gravity. In most flyball governors the second term $\omega^2\int jh\,dM$ is relatively small. In some cases, however, it must be taken into account.

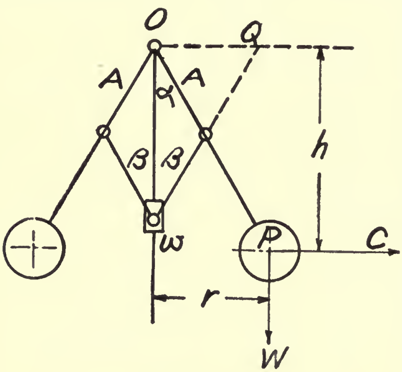


FIG. 203.

128. The Elementary Flyball Governor.—In Fig. 203 is shown the simplest form of flyball governor. The weight W is supposed to be concentrated at the point P. The arms A and B and the slide w are supposed to be weightless. Then if the shaft S revolves with an angular velocity ω, the centrifugal force of the weight W is:

$$C=\frac{W}{g}r\omega^2, \qquad (1)$$

and the moment of this force about O is:

$$T_c=\frac{W}{g}rh\omega^2. \qquad (2)$$

The moment of the weight W about O is:

$$T_w=Wr. \qquad (3)$$

[1] The integral $\int jh\,dM$ is called the product of inertia. For most symmetrical figures the product of inertia is zero.

If $T_w = T_c$ the governor is in equilibrium and therefore,

$$h = g/\omega^2. \qquad (4)$$

The same result can be obtained by the principle of reduced forces as follows: the point Q is the instantaneous center between the arm A and the slide w. Therefore

$$V_p : V_q = OP : OQ. \qquad (5)$$

Hence the force C can be replaced by a vertical force C_0 acting upward at Q, and of such magnitude that

$$C_0 \cdot OQ = C \cdot OP \cos\alpha = Ch = \frac{W}{g} gr\omega^2 h. \qquad (6)$$

The force W can be replaced by a force W_0 acting downward at Q where

$$W_0 \cdot OQ = W \cdot OP \sin\alpha = Wr. \qquad (7)$$

If the governor is in equilibrium $C_0 = W_0$ and therefore

$$Wr = \frac{W}{g} r\omega^2 h \qquad (8)$$

or

$$h = g/\omega^2. \qquad (9)$$

The mass $M = W/g$ can be replaced by a mass M_0 located at Q provided that the relation

$$M_0 V_q^2 = M V_p^2$$

is satisfied. Therefore

$$M_0 = \frac{M OP^2}{OQ^2}. \qquad (10)$$

Since the velocity of the point Q is the same as that of the slide w the forces and mass might equally well have been reduced to any point of the slide w. If the governor is not in equilibrium the slide w will be given an upward acceleration according to the equation

$$A = \frac{C_0 - W_0}{M_0}. \qquad (11)$$

It will be noted that in this type of governor the height h depends solely on the speed ω. The curve Fig. 204 shows the relation between h and ω. In this curve h is given in inches and ω in r.p.m. From the figure it is evident that at high speeds a large change in the speed produces only a very small movement of the governor.

Example.—An elementary flyball governor runs normally at 300 r.p.m. The speed of the engine is now changed to 400 r.p.m. Find the movement of the governor. Solution: from Equation (4)

$$h_1 = g/\omega_1^2 = \frac{32.2 \cdot 12}{\left(\frac{300 \cdot 2\pi}{60}\right)^2} = \begin{matrix}0.38\\ \text{in.}\end{matrix}$$

$$h_2 = g/\omega_2^2 = \frac{32.2 \cdot 12}{\left(\frac{400 \cdot 2\pi}{60}\right)^2} = \begin{matrix}0.21\\ \text{in.}\end{matrix}$$

In other words a change of speed of 100 r.p.m. causes the governor to rise only about $\frac{1}{6}$ inch.

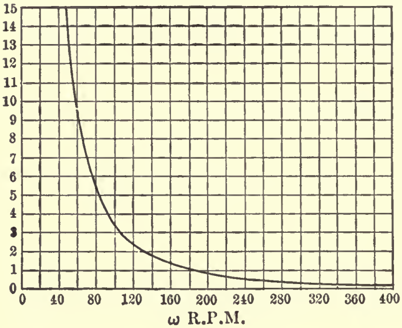


FIG. 204.

On the other hand, if Equation (4) gives a value of h greater than the length OP the governor will not move at all. In this case the weight W hangs vertically against the shaft S. If W be moved out a distance dr from the shaft a centrifugal force is set up such that:

$$dC = \frac{W}{g}\omega^2 dr.$$

The moment of this force about O is:

$$dT_c = \frac{W}{g}\omega^2 OPdr.$$

The weight W has a moment about O given by:

$$dT_w = Wdr.$$

Since

$$g/\omega^2 > OP,\ dT_c < dT_w.$$

Therefore the weight W will simply fall back against the shaft. For example if $OP = 12''$ the minimum speed at which the governor will act is given by the equation,

$$\omega = \sqrt{g/OP} = \sqrt{\frac{32.2 \cdot 12}{12}} = 5.68 \text{ radians per sec.} = 54.3 \text{ r.p.m.}$$

129. Weighted Flyball Governor.—Let the slide w, Fig. 205, have a weight $2bW$. As in the preceding article each of the masses w can be replaced by a mass $M_0 = W/g \left(\frac{OP^2+k^2}{OQ^2}\right)$ moving with the slide w. In this equation k denotes the principal radius of gyration of the arm and weight W. The forces C and W can be replaced by forces C_0 and W_0 acting on w, these forces being given by the equations:

$$C_0 = C\frac{h}{OQ},$$

and

$$W_0 = W\frac{r}{OQ}.$$

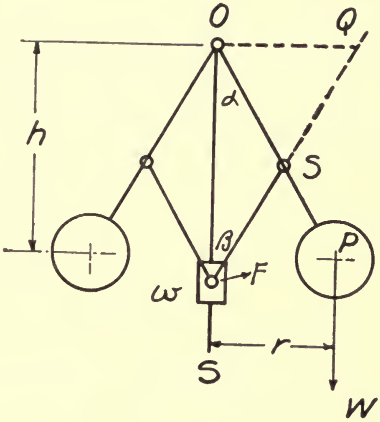


FIG. 205.

For equilibrium

$$2W_0 + 2bW = 2C_0, \quad \ldots \ldots \quad (1)$$

or

$$W\frac{r}{OQ} + bW = \frac{W}{g}\omega^2 r\frac{h}{OQ}. \quad \ldots \ldots \quad (2)$$

Therefore

$$h = g/\omega^2\left(1 + b\frac{OQ}{r}\right). \quad \ldots \ldots \quad (3)$$

In some governors the dimensions are so chosen that $OQ = r$. Then Equation (3) reduces to the simple form:

$$h = g/\omega^2(1+b). \quad \ldots \ldots \quad (4)$$

It is evident that increasing the weight of the slide increases the speed at which the governor will operate in a given position.

Many engines are equipped with a device for varying the weight of the slide and thus controlling the speed of the engine.

A weighted flyball governor will not operate if the speed falls below the value which makes h equal to the length OP. For example if $OP=12''$ and $b=2$ Equation (4) gives:

$$12=\frac{32.2\cdot 12}{\omega^2}(1+2).$$

Therefore $\omega=9.85$ radians per second $=94.1$ r.p.m. is the minimum speed at which the governor will act.

130. Weighted Flyball Governor, Second Type.—In order to make the flyball governor act at low speeds the construc-

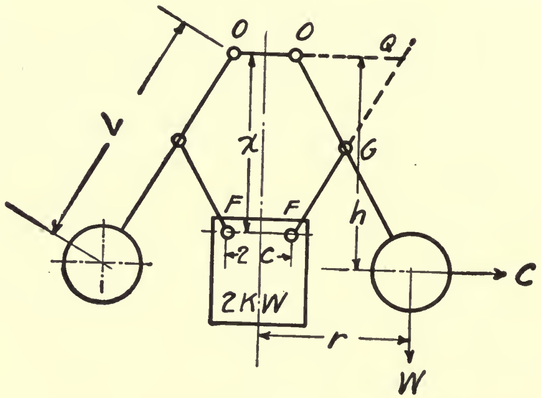


FIG. 206.

tion is slightly changed as indicated in Fig. 206, the points O and the joints F being set at a distance c from the center line of the shaft.

As before, the centrifugal force C and the weight W can be reduced to forces C_0 and W_0 acting on the slide according to the relations:

$$C_0=C\frac{h}{OQ}=\frac{W}{g}r\omega^2\frac{h}{OQ}, \quad \ldots \ldots \quad (1)$$

and

$$W_0=W\frac{r-c}{OQ}. \quad \ldots \ldots \quad (2)$$

Then for equilibrium

$$C_0 = bW + W_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$\frac{W}{g} r\omega^2 \frac{h}{OQ} = bW + W\frac{r-c}{OQ}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Therefore

$$h = g/\omega^2(1 - c/r + b \cdot OQ/r). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

or

$$\omega^2 = g/h(1 - c/r + b \cdot OQ/r). \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When the weights hang vertically downward $r = c$, and $OQ = O$. Therefore $\omega = O$ and the governor operates at all speeds.

131. Horizontal Spring-controlled Governor.—In some governors, particularly for use on small gas engines, the governor spindle is horizontal, and the centrifugal force is opposed by springs instead of by weights. There are two possible arrangements:

(1) The spring may act on the revolving weights, thus directly opposing the centrifugal force.
(2) The spring may act on the slide.

In the first case the spring tension increases uniformly with the distance of the weights from the shaft. That is·

$$S = A + Br, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where S is the spring tension, r the radius, and A and B are constants. The centrifugal force varies directly as the radius. Thus:

$$C = \frac{W}{g} r\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the governor is in equilibrium in a given position r_0 at a given speed ω_0, then

$$A + Br_0 = \frac{W}{g} r_0\omega_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Suppose the weight is now moved an additional distance Δr from the shaft, the speed remaining constant. The centrifugal force now becomes:

$$C + \Delta C = \frac{W}{g}\omega^2(r_0 + \Delta r), \quad . \quad . \quad . \quad . \quad . \quad (4)$$

whence

$$\Delta C = \frac{W}{g}\omega_0^2 \Delta r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The spring tension is also increased by an amount:

$$\Delta S = B\Delta r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If $\Delta S > \Delta C$ the weight will return to its original position and the governor is in stable equilibrium. Comparing (5) and (6)

$$B > \frac{W}{g}\omega_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But $A + Br_0 = \frac{W}{g} r_0 \omega_0^2$, and therefore

$$A < O. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If $\Delta S < \Delta C$ the weight does not return to its original position, but moves on out as far as the construction of the governor allows. Such a governor is said to be *unstable,* and is quite useless for practical purposes.

If $\Delta S = \Delta C$, $A = O$. In this case the weight is in neutral equilibrium in all positions as long as the speed remains constant at ω_0. If he speed increases above ω_0 the weights move out to their extreme positions, and if the speed falls below ω_0 the weights move inward until stopped by the spindle. Such a governor is said to be *isochronous.* Although this represents the ideal governor, which maintains absolutely constant speed under all conditions, it is too sensitive for practical use.

In all practical spring-controlled governors of this type $A < O$. The proper values for the constants A and B are found as follows: let r_1 and r_2 be the maximum and minimum radii permitted by the construction of the governor, and let ω_1 and ω_2 be the maximum and minimum allowable speeds. Then

$$\frac{W}{g} r_1 \omega_1^2 = A + Br_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and

$$\frac{W}{g} r_2 \omega_2^2 = A + Br_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Solving Equations (9) and (10)

$$A=\frac{W}{g}\frac{r_1r_2}{r_1-r_2}(\omega_2{}^2-\omega_1{}^2). \quad . \quad . \quad . \quad . \qquad (11)$$

$$B=\frac{W}{g}\frac{r_1\omega_1{}^2-r_2\omega_2{}^2}{r_1-r_2}. \quad . \quad . \quad . \quad . \qquad (12)$$

The relations between A, B, ω, S, and C are shown in Fig. 207. Here the spring tension S and the centrifugal force C are

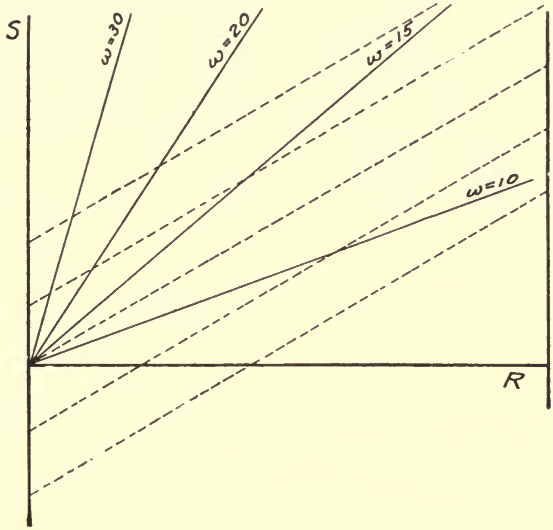


Fig. 207.

plotted as ordinates against the radius r as abscissa. The ordinates of the lines radiating from the origin O, represent the centrifugal forces at different speeds. The dotted parallel lines show the spring tension for a spring of known stiffness B, but with varying initial tension A. The intercepts of the dotted lines on the vertical axis give the initial tensions A. Choosing any one of the dotted lines, the point where it crosses any of the lines of constant speed gives the position in which the governor will be in equilibrium at that speed. Evidently from the figure, if A is positive the speed must be reduced as the governor weight moves out from the center, or in other words the equilibrium is unstable. If A is negative, the higher speed corresponds to the larger radius, and the governor is stable. If A is zero, the line

representing the spring tension passes through the origin, and therefore coincides with one of the lines of constant speed; in other words, the equilibrium is neutral and the governor is isochronous.

Fig. 207 gives values only for a spring having a given stiffness B. If a stiffer spring is used the dotted lines become steeper. Evidently for a spring of a given stiffness and initial tension there is a definite speed above which the governor will not function. This maximum speed corresponds to the last of the constant-speed lines which is intersected by the spring tension line within the limits of movement of the governor.

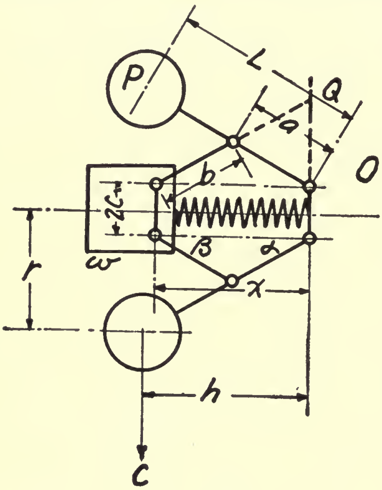


Fig. 208.

132. Horizontal Spring-controlled Governor, Second Type.—In Fig. 208 is shown a horizontal flyball governor controlled by a spring which acts on the slide w in the direction of the center line of the spindle.

The spring tension $=2S=(A-Bx)$. . . . . (1)

The reduced centrifugal force $=C_0=C\dfrac{h}{OQ}$.

The reduced mass $=M_0=M\dfrac{(L^2+k^2)}{OQ^2}$.

For equilibrium $2S=2C_0$.

or

$$A-Bx=C\frac{h}{OQ}=\frac{W}{g}r\omega^2\frac{h}{OQ}. \quad . \quad . \quad . \qquad (2)$$

Therefore

$$h=\frac{g}{W\omega^2}(A-Bx)\frac{OQ}{r}. \quad . \quad . \qquad (3)$$

In most governors of this type $a=b=\frac{1}{2}L$. Then $h=x$ and $OQ=r-c$. Let h_0 be the value of x for which the spring tension is zero. Then

$$S=B(h_0-h). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The reduced centrifugal force becomes:

$$C_0 = C\frac{h}{OQ} = \frac{W}{g}r\omega^2\frac{h}{r-c}. \qquad (5)$$

For equilibrium $C_0 = S$, or

$$\frac{W}{g}\omega^2\frac{rh}{r-c} = B(h_0 - h). \qquad (6)$$

To test the stability of the equilibrium suppose the weights to be moved out a distance dr, the speed ω remaining constant. The slide will be moved a distance dh, and the spring tension will be *increased* by an amount Bdh. The height h is *reduced* by an amount dh. The reduced centrifugal force now becomes:

$$C_0 + dC_0 = \frac{W}{g}\omega^2\frac{(r+dr)(h-dh)}{(r+dr-c)}. \qquad (7)$$

For stability $S+dS > C_0 + dC_0$ or,

$$B(h_0 - h + dh) > \frac{W}{g}\omega^2\frac{(r+dr)(h-dh)}{r+dr-c}. \qquad (8)$$

Clearing of fractions:

$$B(h_0 - h + dh)(r - c + dr) > \frac{W}{g}\omega^2(r+dr)(h-dh) \qquad (9)$$

or

$$B(h_0 - h)(r-c) + B(r-c)dh$$

$$+ B(h_0 - h)dr > \frac{W}{g}\omega^2(rh + hdr - rdh). \qquad (10)$$

Combining (6) and (10) and reducing:

$$(r-c)dh + (h_0 - h)dr > (h_0 - h)(r-c)(dr/r - dh/h). \qquad (11)$$

Collecting terms:

$$(r-c)\frac{h_0}{h}dh + (h_0 - h)\frac{c}{r}dr > 0. \qquad (12)$$

This inequality is evidently satisfied in all cases where $h_0 - h > 0$. In other words the governor is in stable equilibrium provided that the spring tension opposes the outward movement of the balls.

133. Vertical Spring-controlled Governor.—In some vertical governors a spring is arranged to oppose the centrifugal force of

the balls directly, as shown in Fig. 209. There are four forces acting on the mechanism as follows:

(1) Centrifugal force of weights $W = C = \frac{W}{g} r\omega^2$,
(2) Weight of $W = W$,
(3) Tension of spring $= S = (A + Br)$,
(4) Weight of slide $= 2bw$.

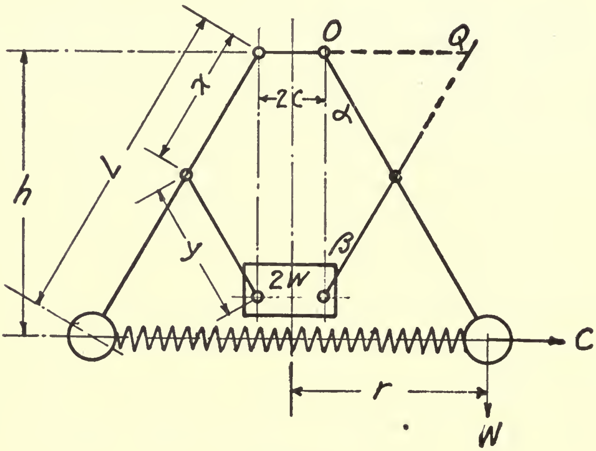


Fig. 209.

All these forces can be reduced to the slide. Considering only one weight and one-half the weight of the slide, the reduced forces are:

$$C_0 = C\frac{h}{OQ} = \frac{W}{g} r\omega^2 \frac{h}{OQ},$$

$$W_0 = \frac{W(r-c)}{OQ},$$

$$S_0 = S\frac{h}{OQ} = (A + Br)\frac{h}{OQ},$$

$$w_0 = w = bW.$$

For equilibrium,

$$C_0 = W_0 + S_0 + w, \quad \dots \dots \dots \quad (1)$$

or

$$\frac{\omega^2}{g} rh = (r - c) + \frac{(A + Br)h}{W} + b \cdot OQ. \quad \dots \quad (2)$$

All the variables in Equation (2) can be expressed as functions of the angle α. In the simplest case where $x=y=\frac{1}{2}L$ Equation (2) reduces to:

$$\frac{L\omega^2}{g}\sin\alpha\cos\alpha=(1+b)\sin\alpha+\frac{A\cos\alpha}{W}+\frac{BL\sin\alpha\cos\alpha}{W}+\frac{Bc\cos\alpha}{W}. \quad (3)$$

or

$$\frac{L\omega^2}{g}=\frac{1+b}{\cos\alpha}+\frac{A}{W\sin\alpha}+\frac{BL}{W}+\frac{Bc}{W\sin\alpha}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To investigate the stability of this governor assume that there are two angles, α_1 and α_2, at which the governor is in equilibrium at the same speed ω, Then

$$\frac{1+b}{\cos\alpha_1}+\frac{A}{W\sin\alpha_1}+\frac{Bc}{W\sin\alpha_1}=\frac{1+b}{\cos\alpha_2}+\frac{A}{W\sin\alpha_2}+\frac{Bc}{W\sin\alpha_2}, \quad . \quad (5)$$

or

$$(1+b)(\sec\alpha_1-\sec\alpha_2)=\left(\frac{A+Bc}{W}\right)(\operatorname{cosec}\alpha_2-\operatorname{cosec}\alpha_1). \quad . \quad (6)$$

If $\alpha_1>\alpha_2$, $\sec\alpha_1-\sec\alpha_2>0$, and $\operatorname{cosec}\alpha_2-\operatorname{cosec}\alpha_1>0$.

Therefore to satisfy Equation (6) $A+Bc>0$. But $A+Bc$ is the spring tension when the weights hang vertically. Consequently the governor will be unstable over some part of its range if the spring is in tension when the weights hang vertically. Example:

Let $\alpha_1=30°$, $\alpha_2=10°$, and $b=1$.

From Equation (4)

$$\frac{L\omega_2}{g}=(1+b)\sec\alpha+\frac{A+Bc}{W}\operatorname{cosec}\alpha+\frac{BL}{W}.$$

The left-hand member of this equation represents the reduced force tending to raise the slide, and the right-hand member the forces opposing this motion. If the left-hand member is larger than the right the governor will rise, and vice versa.

From Equation (6)

$$\frac{A+Bc}{W}=\frac{2(\sec 30°-\sec 10°)}{\operatorname{cosec}10°-\operatorname{cosec}30°}=0.0742.$$

Substituting this value in Equation (4),

$$\frac{L\omega^2}{g}-\frac{BL}{W}=(1+b)\sec 10^\circ+0.0742\operatorname{cosec} 10^\circ$$

$$=(1+b)\sec 30^\circ+0.0742\operatorname{cosec} 30^\circ=2.458.$$

The governor is therefore in equilibrium at the same speed ω in both the 10° and the 30° position. For any intermediate position, say $\alpha=20^\circ$ the right-hand member becomes:

$$(1+b)\sec 20^\circ+0.0742\operatorname{cosec} 20^\circ=2.345.$$

The left-hand member is now larger than the right-hand, which means that the governor will not remain in position at 20°, but will move on out until it comes to rest in its new equilibrium position at 30°.

Spring-controlled vertical governors must therefore be designed with great care so that the region of instability is outside the range of possible positions of the governor. From the inequality (7) it follows that the governor will be stable if the spring is in compression when the weights W hang vertically. It is not usually feasible to use a spring in compression acting directly on the governor weights, but it is possible to design the governor so that in its lowest position the spring will be in tension, although the spring would be compressed if the weights could move downward so far as to hang vertical.

134. Oscillations of Flyball Governors.—In the preceding articles the governors have been considered unstable if there exists more than one equilibrium position for any speed, or if the equilibrium is unstable. In this article a second form of instability is considered. If, on account of a change of speed or for some other reason, the governor is not in its equilibrium position, it will move toward that position. On account of the kinetic energy gained in this motion the governor may go past its equilibrium position and will later return toward it. In this way *oscillations* may be set up which may prevent the governor from properly controlling the speed of the engine. As the governor oscillates on either side of the proper position, the engine is given alternately too much steam and too little, thus causing fluctuations of speed. In some cases such actions may cause serious trouble in the operation of the engine.

As an example consider the weighted governor of Art. 130. The reduced force tending to raise the slide, according to Equation (1), Art. 130, is

$$2C_0 = 2\frac{W}{g} r\omega^2 \frac{h}{OQ}. \quad \dots \dots \dots \quad (1)$$

The reduced forces resisting this motion are

$$\left. \begin{aligned} 2W_0 &= \frac{2W(r-c)}{OQ} \\ \text{and} \qquad 2w &= 2bW \end{aligned} \right\} \quad \dots \dots \dots \quad (2)$$

The reduced mass of the mechanism is

$$2M_0 = \frac{2w}{g} + \frac{2W(L^2+k^2)}{g \cdot OQ^2}. \quad \dots \dots \quad (3)$$

If $C_0 > w + W_0$ the slide will be given an upward acceleration. Let y be the distance of the slide from its equilibrium position. Then

$$\frac{d^2y}{dt^2} = -\frac{C_0 - w - W_0}{M_0}. \quad \dots \dots \dots \quad (4)$$

The oscillations may be studied by means of Equation (4). In the simplest case, where $OG = FG = \frac{1}{2}L$ the geometry of the figure gives the following relations:

$$h = L \cos \alpha = x,$$

$$r - c = OQ = L \sin \alpha,$$

Then

$$C_0 = \frac{W}{g}\omega^2 h \frac{r}{r-c}.$$

$$W_0 = W,$$

$$M_0 = \frac{W}{g}\left(b + \frac{L^2+k^2}{[r-c]^2}\right) = \frac{W}{g}\left(b + \frac{L^2+k^2}{L^2-h^2}\right).$$

Then Equation (4) becomes:

$$\frac{d^2y}{dt^2} = -\frac{r\omega^2 \dfrac{h}{r-c} - (1+b)g}{b + \dfrac{L^2+k^2}{L^2-h^2}}. \quad \dots \dots \dots \dots \quad (5)$$

Let h_0 be the value of h for the position of equilibrium. Then

$$y = x - h_0 = h - h_0,$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2h}{dt^2}.$$

At the position of equilibrium there is no acceleration, or in other words

$$\frac{r_0\omega^2 \dfrac{h_0}{r_0 - c} - (1+b)g}{b + \dfrac{L^2+k^2}{L^2-h_0^2}} = 0. \quad \ldots \ldots \quad (6)$$

Adding Equations (5) and (6), and substituting $L^2 - h^2 = (r-c)^2$,

$$\frac{d^2h}{dt^2} = -\frac{c+\sqrt{L^2-h^2}\, h\omega^2 - (1+b)g}{\dfrac{b(L^2-h^2)+(L^2+k^2)}{\sqrt{L^2-h^2}}} + \frac{c+\sqrt{L^2-h_0^2}\, \omega^2 h_0 - (1+b)g}{\dfrac{b(L^2-h_0^2)+(L^2+k^2)}{\sqrt{L^2-h_0^2}}}. \quad (7)$$

For small oscillations we may write

$$L^2 - h^2 = L^2 - h_0^2.$$

Equation (7) then reduces to

$$\frac{d^2h}{dt^2} = \frac{(h_0 - h)\omega^2}{\dfrac{b(L^2-h_0^2)+(L^2+k^2)}{\sqrt{L^2-h_0^2}}} = \beta^2\,(h_0 - h), \quad \ldots \quad (8)$$

Since

$$(h - h_0) = y, \text{ and } \frac{d^2h}{dt^2} = \frac{d^2y}{dt^2}$$

Equation (8) may be written

$$\frac{d^2y}{dt^2} = -\beta^2 y. \quad \ldots \ldots \quad (9)$$

The general solution of Equation (9) is

$$y = A \cos \beta t + B \sin \beta t, \quad \ldots \ldots \quad (10)$$

where A and B are arbitrary constants. Equation (10) signifies that small oscillations of the governor are *harmonic*, provided (a) that there is no friction, and (b) that the change of speed during the period of oscillation is negligible. Such a governor, there-

fore, if once started oscillating would continue to do so indefinitely. In practical governors there is always friction in the joints which tends to damp out the oscillations, and in many cases an oil dashpot is used to help bring the governor to rest. The resistance of such a dashpot is usually assumed to be proportional to the velocity. If such a device is used Equation (10) becomes:

$$\frac{d^2y}{dt^2} = -\beta^2 y - f^2\frac{dy}{dt}, \quad \text{or} \quad \frac{d^2y}{dt^2} + f^2\frac{dy}{dt} + \beta^2 y = 0, \quad . \quad (11)$$

where f^2 is the resistance of the dashpot at unit speed.

Equation (11) is a linear differential equation whose general solution is

$$y = Ae^{m_1t} + Be^{m_2t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where A and B are arbitrary constants, and m_1 and m_2 are the roots of the quadratic equation

$$m^2 + f^2m + \beta^2m = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

That is

$$m_1 = \frac{-f^2 + \sqrt{f^4 - 4\beta^2}}{2}, \quad \text{and} \quad m_2 = \frac{-f^2 - \sqrt{f^4 - 4\beta^2}}{2}.$$

Substituting the values of m_1 and m_2 in Equation (12)

$$y = e^{-\frac{1}{2}f^2t}\left(Ae^{\frac{1}{2}\sqrt{f^4 - 4\beta^2}} + Be^{-\frac{1}{2}\sqrt{f^4 - 4\beta^2}}\right). \quad . \quad . \quad . \quad (14)$$

If $f^2 > 2\beta$ the roots m_1 and m_2 are real. It is then evident that y approaches 0 as a limit as t indefinitely increased, and that y does not change sign. In other words, the governor moves steadily toward its equilibrium position but does not pass it.

If $f^2 < 2\beta$ the roots m_1 and m_2 are imaginary. Then the general solution becomes

$$y = Ae^{-\frac{1}{2}f^2t} \sin\left(B + \tfrac{1}{2}\sqrt{4\beta^2 - f^4t}\right). \quad . \quad . \quad . \quad . \quad (15)$$

Therefore the governor will oscillate back and forth on either side of the equilibrium position. The oscillations rapidly decrease in magnitude and practically disappear within a very short period of time.

The arbitrary constants in Equations (12), (14) and (15) are found by assuming any set of initial conditions such as, $t=O$, $y=y_0$, and $\frac{dy}{dt}=V_0$. Then from Equation (12)

$$y_0=A+B,$$

$$V_0=m_1A+m_2\beta.$$

From Equation (15)

$$y_0=A \sin \beta,$$

$$V_0=A(\sqrt{4\beta^2-f^4} \cos \beta-f^2 \sin \beta).$$

NOTE.—In the preceding articles two factors have been neglected: (*a*) the effect of the centrifugal force of the link *B*, Fig. 199,(*b*) the term $\int jh\,dM$ in the moment of the centrifugal force (see Article 127). Both these items are of comparatively little importance in most practical governors. For discussion of the effects of these items, with numerical data taken from actual governors, the reader is referred to R. C. H. Heck's "The Steam Engine," Vol. II, page 379 et seq.

135. Effect of Changing Speed of Engine.—Suppose the engine to be running at uniform speed ω_0, and the governor to be in equilibrium with the slide at a height h_0. If the load is now lightened the engine tends to speed up and the governor to rise. Let ω_1 and h_1 represent the new speed of steady running and the new position of equilibrium of the slide. In general it may be said that as long as the governor remains in a position lower than h_1 the speed tends to increase, and that when the governor is above the position h_1 the speed tends to decrease. This tendency is, however, modified by the fact that after cut-off the governor can exert no further influence until the beginning of the next stroke.

When the slide reaches the position h_1 it has a certain velocity V, and will therefore not stop in this position, but will move on to a new point h_2. Thence it will again descend, passing the position h_1 with a speed V'. As the engine decreases in speed during the time which elapses while the slide is moving from h_1 to h_2 and back, the centrifugal force of the weights during the downward movement is less than on the upward movement. In

Curve I, Fig. 210, is shown approximately the manner of variation of this force. In this curve the abscissa is the height h, and the ordinate is the centrifugal force reduced to the slide. Then the area under the curve represents the work performed by the centrifugal force. In the same manner Curve II shows the action of the forces opposing the upward movement of the governor—the weight of the balls and slide, the spring tension, etc. As these forces depend only on the position and not on the speed, Curve II is the same for both the upward and downward movement of the slide. Therefore the work expended in overcoming these forces on the upward movement is returned during the return downward. But the work represented by the area enclosed in the loop of Curve I is not restored, but goes to increase the kinetic energy of the governor. Therefore the velocity V' on the downward stroke will be greater than the velocity V on the upward movement. Consequently the slide will continue to move downward past its original position h_0 to some other point h_3, whence it returns upwards with greater velocity than before. It follows that a frictionless governor will oscillate back and forth with ever-increasing violence until the limits imposed by the construction of the mechanism prevent further increase in the amplitude of oscillation.

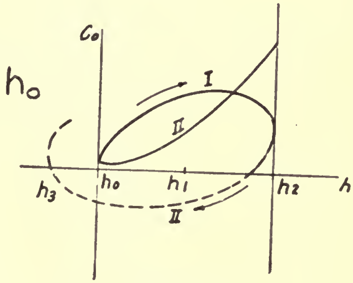


Fig. 210.

If the friction in the mechanism is just sufficient to absorb the work represented by the area enclosed in the loop of Curve I, the amplitude of the oscillations remains constant, and the slide will have an approximately harmonic motion. If the friction is still further increased the oscillations will die out and the governor will come to rest in the new position of equilibrium h_1.

Since excessive mechanical friction is undesirable, such governors are often provided with oil brakes, which quickly damp out the oscillations without impairing the sensitiveness of the governor.

As stated previously these conclusions must be modified on

account of the fact that the governor can act only while steam is being admitted to the cylinder. After cut-off the governor can do nothing until the beginning of the next stroke. This results in greater irregularity of the oscillations, but does not impair the validity of the general conclusions. Heck[1] has attempted to analyze this effect. For further discussion of this point the reader is referred to Heck's treatise.

136. Elementary Centifugal Shaft Governor.—In Fig. 211 is shown an elementary centrifugal shaft governor. The weight W is free to move back and forth in a groove in the flywheel F. This motion is used to adjust the position of the eccentric on the shaft, thus regulating the quantity of steam furnished to the engine. The outward motion of the weight is opposed by the tension of the spring S. If the flywheel rotates with angular velocity ω, the centrifugal force of the weight is

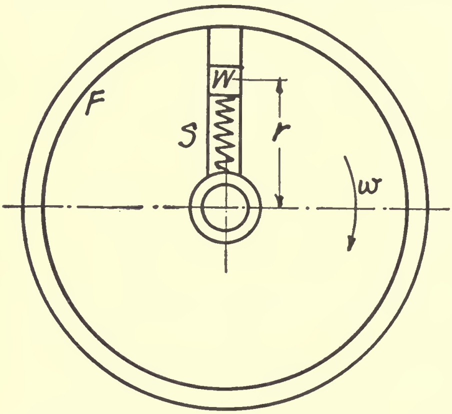


Fig. 211.

$$C = \frac{W}{g} r\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The spring tension increases uniformly with the distance of the weight from the shaft center. That is,

$$S = A + Br.$$

It will be noted that the forces in this governor are precisely the same as those in the governor described in Art. 131. Also the conditions for stability and the method of calculating the proper strength of the spring are the same as those developed in Art. 131.

While a few governors have been made to operate on this

[1] The Steam Engine. Volume II.

plan, a different construction is usually employed and other forces besides the spring tension and centrifugal force are brought into play.

137. Elementary Inertia Governor.—In Fig. 212 is shown an elementary inertia governor. The weight W is pivoted at the center of the shaft and revolves with the flywheel. As long as the flywheel rotates at uniform speed ω there is no relative motion between the weight and the wheel. If, however, the wheel is given an angular acceleration α, the weight, due to its inertia, resists this acceleration, and a moment $I\alpha$ is set up which can be used to shift the eccentric relative to the shaft.

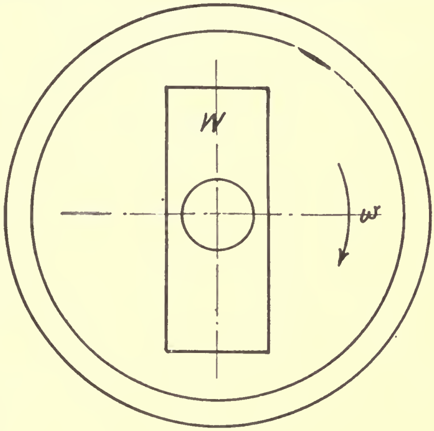


FIG. 212.

This type of governor is more sensitive than the centrifugal governor. It begins to act as soon as the flywheel begins to change its speed, while the centrifugal governor does not move until the increase in centrifugal force $\frac{W}{g}r(\omega_2{}^2 - \omega_1{}^2)$ is great enough to overcome the friction of the mechanism.

The elementary inertia governor cannot be used in practice, since it does not fix the speed at which the engine will run. If the engine is running at any speed ω this governor will regulate at that speed. If, by any means, the speed of the engine is changed to some other value, the governor will regulate equally well at the new speed.

Nearly all practical shaft governors operate by combination of inertia and centifugal force. Governors are classified according to which force predominates.

138. The Inertia Forces.—Consider any mass M, Fig. 213, pivoted at P to an arm of the flywheel. The flywheel rotates about its center O with angular velocity ω and angular acceler-

ation α. The mass M has a rotation about P with angular velocity Ω and angular acceleration $\frac{d\Omega}{dt}$ relative to the wheel. Then by Coriolis' law any point Q of the mass has the following accelerations:

(1) $S\omega^2$ in the direction QO,
(2) $S\alpha$ at right angles to QO,
(3) $x\Omega^2$ in the direction QP,
(4) $x\frac{d\Omega}{dt}$ at right angles to QP,
(5) $2u\omega = 2x\omega\Omega$ along QP.

If an elementary mass dM is concentrated at Q, this mass must be subject to forces equal to dM multiplied by these various accelerations.

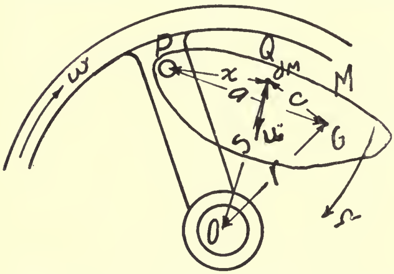


FIG. 213.

The force $S\omega^2 dM$ may be broken up into two components $r\omega^2 dM$ and $c\omega^2 dM$ parallel to GO and QG, respectively, where G is the center of gravity of the mass M. The sum of all the elementary forces $r\omega^2 dM$ is $r\omega^2 M$ and passes through G. The sum of all the elementary forces $c\omega^2 dM$ is

$$\omega^2 \int c\, dM = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Both these results follow directly from the definition of the center of gravity. The entire system of forces of the form $S\omega^2 dM$ can then be replaced by a single force $Mr\omega^2$ acting through G in the direction GO.

Similarly all the elementary force of the form $S\alpha dM$ can be resolved into components $r\alpha dM$ and $c\alpha dM$ at right angles to OG and GQ respectively. This resolution is shown in Fig. 214.

The resultant of all the forces $r\alpha dM$ is a single force $r\alpha M$ acting through G at right angles to OG. The summation of the forces $c\alpha dM$ is given by the expression

$$\alpha \int c\, dM = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It should be noted, however, that, though the sum of these forces is zero, their moments about G do not vanish. The moment of $c\alpha dM$ about G is $c^2\alpha dM$. The sum of the moments of all these forces is

$$\int c^2\,\alpha dM = I\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where I is the moment of inertia of the mass M about G.

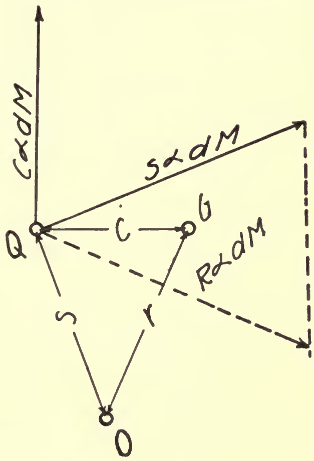


FIG. 214.

The forces $x\Omega^2 dM$ can be combined into a single resultant $a\Omega^2 M$ acting through G in the direction GP. The proof is exactly the same as for the forces of the form $S\omega^2 dM$.

The forces of the form $x\frac{d\Omega}{dt}\,dM$ can be combined into a single force $Ma\,\frac{d\Omega}{dt}$ passing through G in a direction perpendicular to PG, and a couple $I\,\frac{d\Omega}{dt}$. The proof is the same as for the forces of the form $S\alpha dM$. The forces $2x\omega\Omega dM$ can be combined into a single resultant $2M\omega\Omega a$ acting through G along the line PG. The proof is the same as for the forces of the forms $S\omega^2 dM$ and $x\Omega^2 dM$.

The entire system of forces is then reduced to the following:

(a) A single force $Mr\omega^2$ in the direction GO.

(b) A single force $Ma\Omega^2$ in the direction GP.

(c) A single force $Mr\cdot\alpha$ at right angles to GO.

(d) A single force $Ma\,\frac{d\Omega}{dt}$ at right angles to GP.

(e) A single force $2Ma\omega\Omega$ along the line GP.

(f) A couple $I\alpha$.

(g) A couple $I\,\frac{d\Omega}{dt}$.

(h) The spring tension $A+Br$ reduced to the center of gravity.

The six forces all act through the center of gravity G. Forces (b) and (e) pass through P and consequently have no tendency to turn the mass about P. They can therefore be dropped from further consideration. As $\frac{d\Omega}{dt}$ is usually small the force (d) and the couple (g) are often ignored in governor calculations.

The inertia forces are of course equal and opposite to the accelerating forces.

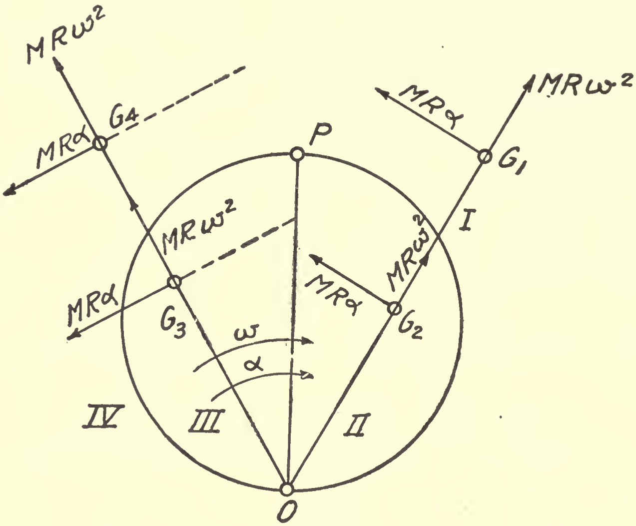


FIG. 215.

The two forces $Mr\omega^2$ and $Mr\alpha$, and the couple $I\alpha$ may all tend to turn the mass M in the same sense about the pivot P or the moment of one may oppose the other two. The relative positions of the points P, O, and G determine the sense in which the different moments act. With the line OP, Fig. 215, as the diameter construct a circle, and extend the diameter OP indefinitely. The plane is thus divided into four regions I, II, III, and IV as shown. Suppose the rotation ω and the acceleration α to be clockwise. The sense of the moments of the forces $Mr\omega^2$ and $Mr\alpha$ about P can be seen by placing the center of gravity G successively at G_1, G_2, G_3, and G_4 in the regions I, II, III, and IV

respectively. Calling counter-clockwise moments positive the results may be shown in compact form in the following table:

Region	I	II	III	IV
$I\alpha$	+	+	+	+
$Mr\omega^2$	+	+	−	−
$Mr\alpha$	+	−	+	−

It is evident that the governing action is most powerful when point G lies in region I. Regions III and IV can be used only when I is small, as in these regions the centrifugal force opposes the inertia couple $I\alpha$. If the couple $I\alpha$ is large the immediate effect of increasing the speed might be to increase the supply of steam furnished.[1]

139. Action of the Governor.—When the engine runs at unifrom speed ω the forces $Mr\alpha$ and the couple $I\alpha$ disappear, and there remains only the centrifugal force $Mr_0\omega^2$. This must be balanced by the spring tension $A+Br_0$. Then if the load on the engine changes the speed tends to change also and an angular acceleration α is given to the flywheel. On account of the increased speed the centrifugal force becomes greater than the spring tension and the governor therefore tends to rotate about the pivot P. On account of the angular acceleration α the force $Mr\alpha$ and the couple $I\alpha$ are also set up and help to cause rotation of the governor weight about P. As soon as the governor starts to move, the force $Ma\dfrac{d\Omega}{dt}$ and the couple $I\dfrac{d\Omega}{dt}$ come into play. These usually tend to retard the motion. Under the influence of all these forces the governor tends to move to a new position.

There is in general only one position at which the governor will be in equilibrium under the new load, and one speed at which the engine will run steadily. Let r_1 be the distance of the center of gravity from the shaft center in this position, and let ω_1 be the new speed of steady running. Then when the governor again reaches equilibrium,

$$Mr_1\omega_1^2=A+Br_1.$$

[1] Stodola. The Siemens Governing Principle. Zeitschrift des Vereines deutscher Ingenieure, 1899.

Before reaching this new condition of equilibrium the governor may, however, oscillate back and forth on either side of the position. Friction in the mechanism tends to damp out these oscillations, but if the friction is small, the governor may under certain conditions continue to vibrate back and forth for a considerable time, thus causing very irregular running.

140. Mathematical Analysis of Governor Action.—The mathematical study of the action of the inertia governor involves great difficulties, and it may be doubted whether a general solution is possible. Suppose that after a time t the engine is running at a speed ω and that the governor has reached a position where the center of gravity is at a distance r from the center of the shaft. Then taking moments about P there results:

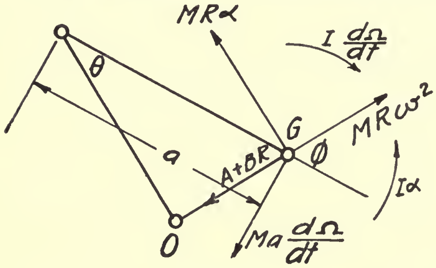


FIG. 216.

$$\text{Moment of } Mr\omega^2 = Mr\omega^2 a \sin\phi \text{ (Fig. 216)}. \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{Moment of } Mr\alpha = Mr\alpha a \cos\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{Moment of } Ma\frac{d\Omega}{dt} = -Ma^2\frac{d\Omega}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{Moment of } I\alpha = I\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\text{Moment of } I\frac{d\Omega}{dt} = -I\frac{d\Omega}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{Moment of spring tension} = -(A+Br)\, a \sin\phi. \quad . \quad . \quad (6)$$

In general the acceleration α is approximately proportional to the distance of the governor from its equilibrium position. Or

$$\alpha = C(r_1 - r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting this value and writing $\Omega=\frac{d\theta}{dt}$, we get

$$\text{Total moment}=Mr\omega^2a\sin\phi+MarC(r_1-r)\cos\phi-Ma^2\frac{d^2\theta}{dt^2}$$

$$+IC(r_1-r)-I\frac{d^2\theta}{dt^2}-(A+Br)\,a\sin\phi=T. \quad . \quad . \quad . \quad (8)$$

This moment tends to cause rotation about P.
Then

$$T=(Ma^2+I)\frac{d^2\theta}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Combining (8) and (9), the resulting equation is

$$2(Ma^2+I)\frac{d^2\theta}{dt^2}=Mr\omega^2a\sin\phi+MarC(r_1-r)\cos\phi$$

$$+IC(r_1-r)-(A+Br)\,a\sin\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also

$$\omega=\omega_0+\int_0^t\alpha\,dt=\omega_0+C\int_0^t(r_1-r)dr. \quad . \quad . \quad . \quad 11)$$

Since ϕ and r can be expressed as functions of θ, Equation (10) reduces to a single differential equation between θ and t. Unfortunately, this equation does not admit of any general solution. It is possible, however, to draw some general conclusions from the investigation. Suppose the engine to be running steadily at a speed ω_0, and the governor to be in equilibrium at a position defined by the angle θ_0. Now let the load be suddenly lightened. Let the new speed of steady running be ω_1 and the new position of equilibrium of the governor be defined by the angle θ_1. The governor at once begins to move towards its new position of equilibrium, and the engine begins to speed up. By the time the governor reaches the position θ_1 the speed of the engine will have *increased* to some value ω which may be *greater or less* than ω_1. The governor will have acquired a certain angular velocity Ω about the pivot P, and will therefore continue to move outward beyond θ_1 to some new position θ_2. After the governor passes θ_1 the speed of the engine begins to diminish and the centrifugal force consequently tends to grow less. After the governor has reached its extreme position θ_2 it starts to return toward its original position. As the speed of the engine decreases after the governor

passes θ_1, the centrifugal force is less during the return movement of the governor than during its outward travel. In Curve I, Fig. 217, is shown the moment of the centrifugal force about the pivot P. Since the ordinate of this diagram represents moment about P and the abscissa represents angular movement about the same point, the area under the curve represents the work done by the centrifugal force during the motion of the governor. In like manner Curves II, III, and IV represent the moments of the

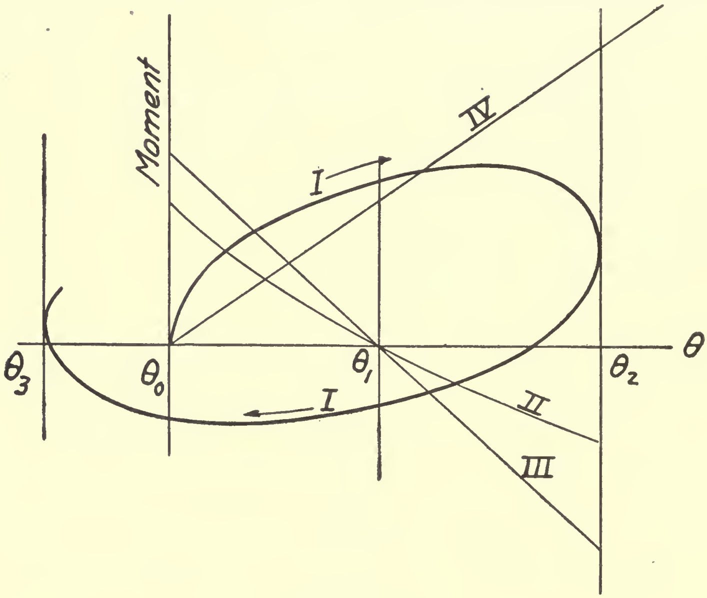


Fig. 217.

inertia force $Mr\alpha$, the inertia couple $I\alpha$ and the spring tension $A+Br$, respectively. Since these forces are independent of the speed and are functions of θ only, the curves are the same whether the governor is traveling from θ_0 toward θ_2 or vice versa. Therefore the work done by these forces in accelerating the motion of the governor during its outward movement is absorbed in retarding the governor during its return over the same path. But the work represented by the area enclosed in the loop of Curve I is not thus reabsorbed, and this work serves to increase the kinetic energy of the governor. Therefore the governor returns at a

higher speed than it had on its outward travel. Consequently it will pass its original position θ_0 and go on to another position θ_3, whence it again moves outward with greater velocity than before. In other words, a frictionless governor will move back and forth with increasing amplitude of oscillations until a limit is reached which is fixed by the construction of the mechanism.

Friction in the mechanism tends to reduce the speed of the governor's motion and to diminish the amplitude of the oscillations. Static friction in the joints, etc., opposes the motion with a nearly constant resistance. If this resistance is sufficient to absorb the work represented by the area enclosed in the loop of Curve I, the amplitude of the oscillations remains constant, and the motion of the governor is approximately harmonic. If the friction is further increased, the oscillations gradually diminish until the governor comes to rest in its new position of equilibrium.

Since excessive friction is objectionable on account of its tendency to hinder the prompt adjustment of the governor to small changes of load, it is customary to provide this type of governor with an oil brake. The resistance of this brake is approximately proportional to the velocity. With this device the sensitiveness of the governor is not impaired, while large and violent oscillations are quickly damped out. For more complete discussion of the oscillations in shaft governors, the reader is referred to the article by Stodola in the Zeitschrift des Vereines deutscher Ingenieure, May, 1899.

The friction and inertia of the eccentric, valve rod, etc., also deserve consideration. For a discussion of these points the reader is referred to Lanza's " Dynamics of Machinery."

CHAPTER VIII

THE MECHANICS OF THE GYROSCOPE

141. Introductory.—One of the most interesting examples of inertia forces and kinetic reactions is found in the *gyroscope.* Inertia forces in general and the kinetic reactions produced by them may be very undesirable, and under extreme conditions even dangerous, but in some cases they may be made very useful in accomplishing desired results. This is especially true in gyroscopic motion.

Recent applications of the gyroscope to engineering problems, especially its use in the stabilizing of ships, has given fresh interest to what for years has been considered mainly as a toy for children or as a problem to test the ability of mathematicians.

142. Gyroscopic Couple.—The gyroscope here considered will be a rigid body, symmetrical with respect to three intersecting axes mutually at right angles, the body rotating or spinning about one of the axes with freedom to turn about one or both of the other two axes as the case may be.

Referring to Fig. 218, let OZ, OX, and OY be three intersecting axes mutually at right angles, $ABCD$ being a horizontal circular disk. Let the disk rotate or spin with a high constant angular velocity, ω, about the Z axis and at the same time let the disk turn about the Y axis with an angular velocity Ω. A particle, m, at the distance ρ (assumed at the circumference for convenience) from the center, O, will have a constant velocity, $\rho\omega$ in the plane of the disk. It will also have a velocity perpendicular to the plane of the disk (parallel to Z axis) due to the angular velocity Ω. This component of the total velocity, in one revolution of the disk about the Z axis, will change in magnitude from zero at A increasing downward to a maximum at B, then decreasing to zero at C, then increasing (in opposite direction) to a maximum at

D and finally decreasing to zero at A again. The magnitude of this velocity is $\Omega\rho \cos \theta$.

As stated above, a force is always required to change the magnitude of a velocity, the direction of the force being in the direction of the change in velocity. Hence a force must be acting downward on the particles which are traveling from D to B and upward on particles traveling from B to D. The velocity $\omega\rho$ may be replaced by two components, one perpendicular to OY, $\omega\rho \sin \theta$, and one parallel to OY, $\omega\rho \cos \theta$. The component, $\omega\rho \cos \theta$, may be neglected as far as rotation about OY is con-

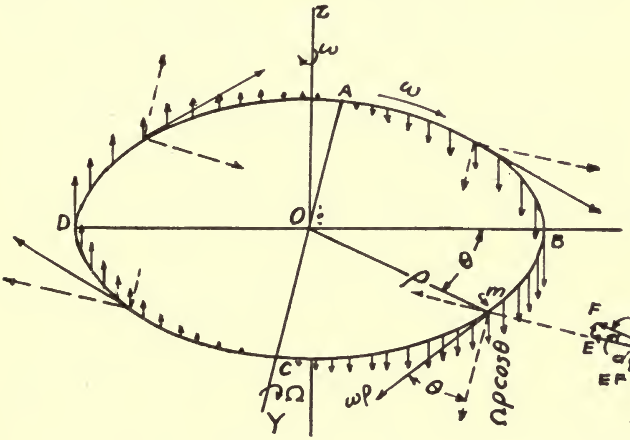


Fig. 218.

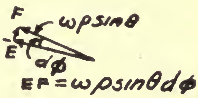


Fig. 219.

cerned since the force causing a change in this component has no moment with respect to OY. The component $\omega\rho \sin \theta$ suffers a change in direction due to the angular velocity Ω. A force is required to change the direction of the velocity of a particle, the direction of this force being the same as the direction of the change in direction of the velocity. It is evident from the figure that this force acts downward on particles which are traveling from D to B and upward on particles traveling from B to D. The disk, therefore, due to these two forces acting on each particle, would start to rotate about the X axis unless a couple acted to prevent the turning. This couple is called the *Gyroscopic Couple.*

One of the simplest physical illustrations of this gyroscopic reaction is obtained by holding a bicycle wheel with one hand

on either end of the projecting axle (dismounted from frame). If the wheel is spinning at a high velocity in a vertical plane, an attempt to turn the axle (and hands) in the horizontal plane will meet with resistance and the wheel will start to turn in the vertical plane unless the hands exert a couple to prevent this turning.

In order to determine the magnitude of the gyroscopic couple proceed as follows: The component of the acceleration of any particle, m, parallel to OZ is $a=a_1+a_2$ where a_1 represents the rate of change of the magnitude of the velocity $\Omega\rho\cos\theta$ and a_2 represents the rate of change of direction of the velocity $\omega\rho\sin\theta$.

$$a_1=\frac{d(\Omega\rho\cos\theta)}{dt}=\Omega\rho\sin\theta\frac{d\theta}{dt}=\Omega\rho\sin\theta\omega=\omega\Omega y. \quad . \quad . \quad (1)$$

EF, Fig. 219, represents the change in direction of the velocity $\omega\rho\sin\theta$ during an interval of time dt while the disk turns through an angle $d\theta$ about OY. Hence, remembering that an arc equals the subtended angle times the radius and that the sine of a small angle equals the angle itself, we have:

$$a_2=\frac{EF}{dt}=\frac{\omega\rho\sin\theta\, d\phi}{dt}=\omega\rho\sin\theta\Omega=\omega\Omega y, \quad . \quad . \quad (2)$$

therefore

$$a=2\omega\Omega y,^1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the force then acting on the particle of mass m, perpendicular to the plane of the disk producing the acceleration a, is

$$F=ma=m2\omega\Omega\,\rho\sin\theta=m2\omega\Omega y. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The moment of this force with respect to the X axis is

$$dT=Fy=2m\omega\Omega y^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the total gyroscopic couple is

$$T=\int dT=\int 2m\omega\Omega y^2=2I_x\omega\Omega, \quad . \quad . \quad . \quad . \quad (6)$$

but since the polar moment of inertia (with respect to Z axis) is $I=2I_x$ we have

$$T=I\omega\Omega.$$

[1] It should be noted that this result could have been obtained with much less work by applying Coriolis' law.

The following conclusion may be drawn: If a rotating or spinning body is turned about an axis perpendicular to the axis of spin a couple will be set up about an axis perpendicular to the other two axes. The magnitude of the couple equals the product of the angular momentum of spin times the angular velocity of turning. Or, in other words, a body spinning about a Z axis with angular velocity ω, requires a couple $C = I\omega\Omega$ to maintain an angular velocity Ω about the Y axis. Ω is called the velocity of precession, or the body is said to precess with an angular velocity Ω.

143. Surging.—Attention should be called to the fact that $C = I\omega\Omega$ is the couple necessary to maintain a precession which has already been started. Referring to Fig. 220, during the time which elapses while the angular velocity Ω is being set up the wheel alternately dips and rises. While the wheel is descending the angular velocity Ω increases, and while the wheel is rising Ω decreases. This action is known as *surging.* Usually friction soon causes the oscillations to die out, and the wheel continues to precess at nearly constant speed.

Suppose the shaft of the wheel Fig. 220, to be supported in a horizontal position, and the wheel to be given an angular velocity ω about the axis $O'Z$. Now if the support is removed from the end of the shaft the wheel starts to drop, and thus a rotation about the axis $O'X'$ is set up. If θ is the angle through which the wheel turns about the axis $O'X'$, the angular velocity of the rotation about $O'X'$ is $\frac{d\theta}{dt}$. As the wheel now has simultaneous rotation about the axes OZ and $O'X'$ a gyroscopic couple is induced which tends to produce rotation in a horizontal plane about the axis $O'Y$. The magnitude of this couple is $I\omega\frac{d\theta}{dt}$. As the end of the shaft is free to move, the wheel is given an angular velocity Ω about $O'Y$, and this rotation together with the rotation ω produces a gyroscopic couple of magnitude $I\omega\Omega$, which tends to cause the wheel to rise.

After an interval of time t from the beginning of the surging action, let

$$\theta = \text{angle of dip},$$

and

$$\Omega = \text{angular velocity about } O'Y.$$

Then the moment of the gyroscopic couple tending to right the wheel is $I\omega\Omega$, and the moment tending to cause rotation about $O'X'$ is

$$Wl \cos\theta - I\omega\Omega.$$

For small values of θ, $\cos\theta$ is approximately 1. Therefore the moment about $O'X'$ is $Wl - I\omega\Omega$. This moment causes an angular acceleration about $O'X'$ such that, neglecting friction,

$$Wl - I\omega\Omega = \left(I_x + \frac{W}{g}l^2\right)\frac{d^2\theta}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

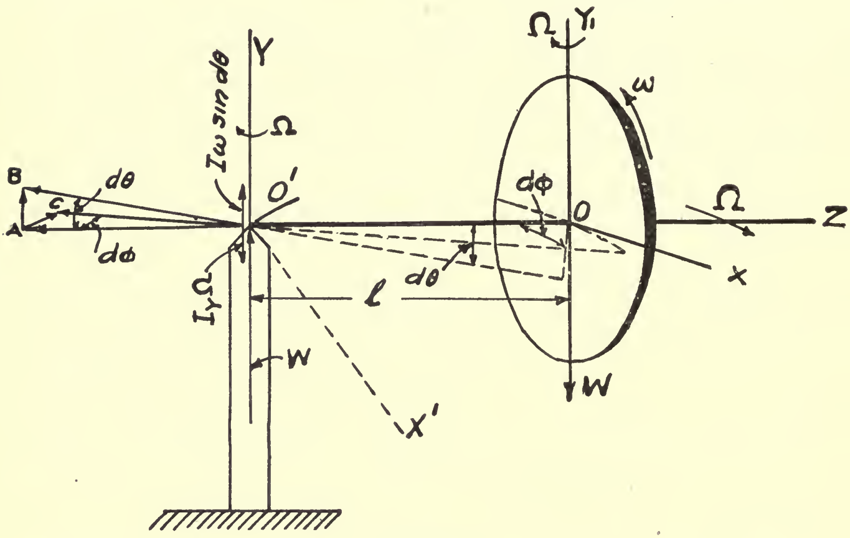


Fig. 220.

The gyroscopic couple $I\omega\frac{d\theta}{dt}$ produces an angular acceleration $\frac{d\Omega}{dt}$ about $O'Y$ such that

$$I\omega\frac{d\theta}{dt} = \left(I_x + \frac{W}{g}l^2\right)\frac{d\Omega}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The surging action can be studied by means of Equations (1) and (2). Differentiating Equation (2),

$$\frac{d^2\theta}{dt^2} = \frac{I_x + \frac{W}{g}l^2}{I\omega}\,\frac{d^2\Omega}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Combining Equations (1) and (3),

$$Wl - I\omega\Omega = \frac{\left(I_x + \frac{W}{g}l^2\right)^2}{I\omega}\frac{d^2\Omega}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Equation (4) may be written in the form

$$\frac{d^2\Omega}{dt^2} + \left(\frac{I\omega}{I_x + \frac{W}{g}l^2}\right)^2 \Omega = \frac{WlI\omega}{\left(I_x + \frac{W}{g}l^2\right)}. \quad . \quad . \quad . \quad (5)$$

or

$$\frac{d^2\Omega}{dt^2} + A^2\Omega = B^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The general solution of Equation (6) is

$$\Omega = C_1 \sin At + C_2 \cos At + \frac{B^2}{A^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

At the instant $t=0$, $\Omega=0$ and $\frac{d\theta}{dt}=0$. From Equation (2) if $\frac{d\theta}{dt}=0$, $\frac{d\Omega}{dt}=0$. Therefore

$$C_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$C_2 = -\frac{B^2}{A^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Equation (7) then reduces to

$$\Omega = \frac{B^2}{A^2}(1 - \cos At). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

From Equation (2)

$$\frac{d\theta}{dt} = \frac{1}{A}\frac{d\Omega}{dt} = \frac{B^2}{A^2}\sin At. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The interpretation of Equations (10) and (11) is that neglecting friction, the velocities of rotation about $O'X$ and $O'Y$ follow the law of harmonic oscillation.

If friction is taken into account, Equation (6) must be modified by the addition of a term which takes account of the friction. Let it be assumed that the friction is proportional to the velocity $\frac{d\theta}{dt}$. From Equation (2) it follows that the friction is also proportional to $\frac{d\Omega}{dt}$. Equation (6) then becomes

$$\frac{d^2\Omega}{dt^2} + F^2\frac{d\Omega}{dt} + A^2\Omega = B^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The general solution of Equation (12) is

$$\Omega=C_1e^{m_1t}+C_2e^{m_2t}+\frac{B^2}{A^2}, \quad \dots\dots \quad (13)$$

where m_1 and m_2 are the roots of the auxiliary equation

$$m^2+F^2m+A^2=0. \quad \dots\dots\dots \quad (14)$$

Therefore

$$m_1=\frac{-F^2+\sqrt{F^4-4A^2}}{2},$$

and

$$m_2=\frac{-F^2-\sqrt{F^4-4A^2}}{2}.$$

To determine the constants C_1 and C_2 put $\Omega=0$, and $\frac{d\Omega}{dt}=0$, when $t=0$. Then from Equation (13)

$$C_1+C_2+\frac{B^2}{A^2}=0,$$

and

$$m_1C_1+m_2C_2=0.$$

Therefore

$$C_1=\frac{m_2}{m_1-m_2}\frac{B^2}{A^2},$$

and

$$C_2=-\frac{m_1}{m_1-m_2}\frac{B^2}{A^2}.$$

If $F^2>2A$ the roots m_1 and m_2 are real, In this case Ω increases steadily to a maximum value of $\frac{B^2}{A^2}=\frac{Wl}{I\omega}$, and thereafter remains constant. If $F^2<2A$ the roots m_1 and m_2 are imaginary. In this case the general solution of Equation (13) is

$$\Omega=(C_3\sin At+C_4\cos At)e^{-\frac{1}{2}F^2t}+\frac{B^2}{A^2}. \quad \dots \quad (16)$$

The interpretation of Equation (16) is that Ω approaches the value $\frac{B^2}{A^2}$ as a limit, after oscillating back and forth on both sides of this value. In other words the phenomenon of surging is present, but the oscillations are soon damped out, and thereafter the wheel continues to precess at constant speed.

144. Line of Action of Resultant.—From the equation $a = 2\omega\Omega y$ it is noted that all points at the same distance, y, from the X axis have the same acceleration perpendicular to the disk. The resultant force acting (perpendicular to disk) on each half as divided by the X axis may be found by summing up the forces $F = 2m\omega\Omega y$ for each half, which leads to the equation,

$$R_z = \frac{M}{2}a_z = \frac{M}{2}\cdot 2\omega\Omega y = M\omega\Omega\frac{4r}{3\pi}.$$

The action line of each of these two resultants, R_z, passes through a point at a distance y_0 from the center along the Y axis as found from the moment equation

$$I\omega\Omega = M\omega\Omega\frac{4r}{3\pi}2y_0$$

but since $I = \frac{1}{2}Mr^2$ (for disk)

$$y_0 = \frac{3\pi r}{16} = 0.59r.$$

145. Wheel on Circular Path.—A disk or wheel rolling on a circular track is an illustration of gyroscopic motion. Many other examples of such motion will occur to the reader, such as locomotive drivers or automobile wheels and flywheels when rounding a curve, etc. A couple $C = I\omega\Omega$ is required to keep the disk from overturning, or in other words the gyroscopic couple of the drivers helps the centrifugal couple to overturn the locomotive. It is interesting to note that the line of zero acceleration perpendicular to the disk is at $y = -\frac{r}{2}$ as shown as follows: Referring to Fig. 221, the velocity of any point (assumed at the circumference for convenience) is made up of three components as shown. The component of the acceleration of this point which is parallel to the Z axis is $a = a_1 + a_2 + a_3$. a_1 and a_2 have the same meaning as in the case already considered $a_1 = a_2 = \omega\Omega y$. $a_3 = \Omega^2 R$ due to the rate of change of the direction of ΩR. Hence,

$$a = 2\omega\Omega y + \Omega^2 R.$$

The velocity of the center of the disk is equal to the velocity of rim or $\Omega R = \omega r$. Equating a to zero and solving for y we get,

$$y = -\frac{\Omega^2 R}{2\omega\Omega} = -\frac{r}{2},$$

that is, all points on a line parallel to the X diameter midway between the track and the X diameter have no acceleration parallel to the Z axis. All points above this line have accelerations directed toward the inside of the track and all points below this line have accelerations directed outward, the acceleration in each case being proportional to the distance from this line. The resultant of the forces producing these accelerations may easily be found. This leads to the fact that two equal masses fixed along any diameter at the same distance from the center of the

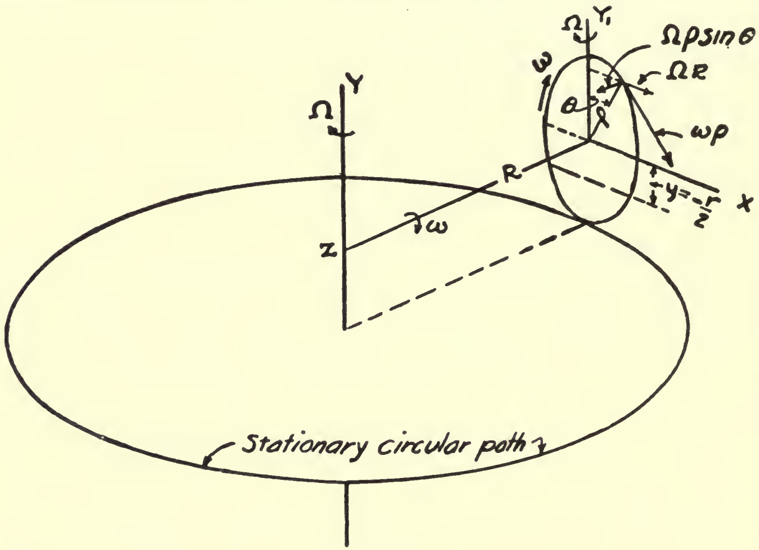


FIG. 221.

disk would cause no uneven pressure on the track when running on a straight track and would cause variation in pressure when running on a curved track. It is an interesting fact easily proved that three or more masses arranged symmetrically with respect to OZ will cause no variation in pressure on either track. The variation in pressure in the case of a locomotive driver is very small even when running at high speed around a curve of the smallest radius.

146. Angular Momentum.—As stated above, when an external couple acts on a rigid body the angular momentum of the body,

$I\omega$, will change, the magnitude of the couple being measured by the rate at which the angular momentum changes. The three properties of an angular momentum which are involved in its physical meaning are (1) magnitude, (2) sense of rotation, (3) direction (not location) of the plane of rotation, as specified by the angle which the plane of rotation or its normal makes with some fixed plane (or line). A vector may fully represent these three factors by drawing it of a proper length to some scale perpendicular to the plane of rotation with the arrow pointing in the direction in which a right-handed screw would advance if turned in the direction of rotation. Any change in the angular momentum is fully represented by the corresponding change in the vector. The couple required to produce this change is always in a plane perpendicular to the vector representing the rate of change in the angular momentum.

It is evident that the rate of change of $I\omega$ may be (1) the rate of change of the magnitude only, as given by $I\frac{d\omega}{dt}=I\alpha$ or (2) the rate of change of the direction of the $I\omega$ vector only or (3) the rate of change of both magnitude and direction.

The principle of conservation of angular momentum states that if no external couple acts in a given plane on a system of particles, the angular momentum of the system with respect to an axis perpendicular to the plane remains unchanged.

147. Illustrations.—A few familiar illustrations may serve to make clear the physical meaning of the principles mentioned above. Let two pulleys be mounted on a rotating shaft, one keyed to the shaft, and rotating with it, the other loose (not rotating). Let a clutch operate suddenly to throw in the loose pulley, thus starting it to rotate with the tight pulley and axle. The angular momentum of the whole system before the clutch was operated was $I_1\omega_1$. By throwing in the clutch the moment of inertia of the rotating system was increased to (I_1+I_2), while the angular velocity decreased to ω_2. But from the principle of conservation of angular momentum we get,

$$I_1\omega_1=(I_1+I_2)\omega_2 \text{ or } \omega_2=\frac{I_1}{I_1+I_2}\omega_1.$$

The angular velocity ω_2 is reached after a surging action dies out. While these vibrations are being extinguished kinetic

energy is lost. The kinetic energy of the whole system before the clutch was operated was $\frac{1}{2}I_1\omega_1^2$, and after the two pulleys have come to a constant velocity, ω_2, the kinetic energy of the system is

$$\tfrac{1}{2}(I_1+I_2)\omega_2^2=\tfrac{1}{2}(I_1+I_2)\frac{I_1^2}{(I_1+I_2)^2}\omega_1^2=\tfrac{1}{2}I_1\omega_1^2\left\{\frac{I_1^2}{I_1+I_2}\right\},$$

hence

$$\left\{1-\frac{I_1}{I_1+I_2}\right\},$$

represents the part of the original kinetic energy which is dissipated. This is analogous to the loss of energy in setting up precession in the gyroscope.

A man if standing on a platform which is revolving with an angular velocity ω can decrease the ω by extending his arms thus increasing his moment of inertia. This develops a friction couple on the soles of his shoes but the product $I\omega$ for the man and platform must remain constant or if ω is to be kept constant an external couple C must be applied to the platform such that

$$C=\omega\frac{dI}{dt}.$$

A gymnast leaving the swinging trapeze at the top of a circus tent with angular velocity ω, his body being extended, may make several complete turns in mid-air as he descends in the vertical plane by "doubling up." This decreases his moment of inertia and gives him a corresponding increase in angular velocity, the product $I\omega$ remaining constant, since no couple acts in the vertical plane.

As is well known, a cat when held by the four feet will, if dropped, always turn through one-half a revolution and light on its four feet (providing it is held a reasonable distance from the floor). This is easily explained by the principle of conservation of angular momentum. The $I\omega$ of the cat with respect to an axis corresponding to its long dimension is zero when its feet are released, and must remain zero during its fall, since no couple acts to change this angular momentum. For small relative displacements the cat may be considered as made up of a fore and hind part with a swivel at the center. Let I_1 and I_2 represent

the moment of inertia of the fore and hind parts respectively. By extending the hind legs and contracting the fore legs the fore part is turned in one direction, throwing the hind part in the opposite direction. Since, however, $I_2 > I_1$ there is a net gain in favor of the fore part. Now by extending the fore legs and contracting the hind legs the hind part is turned, throwing the fore part back but to a less extent, since now $I_1 > I_2$. By a rapid succession of such actions and reactions the cat makes one-half a turn. The reader may, by the same method, turn through any desired angle, when sitting on a swiveled chair, by extending his legs and contracting his arms, etc., etc. The action is more positive if weights are held in the hands.

In the above illustrations the change in the angular momentum was either zero or a change in magnitude only. That is, the plane of rotation remained the same, or in other words, the vector representing $I\omega$ changed in length only. In gyroscopic motion the $I\omega$ vector changes direction.

148. Analysis of Gyroscope by Method of Angular Momentum.—As has been stated, a disk rolling around a curved path is revolving simultaneously about two axes mutually at right angles. This example of gyroscopic motion together with the case as shown in Fig. 220 will be used (for convenience) to show the applications of the foregoing principles concerning angular momentum.

Referring to Fig. 222 let A_1B_1 represent the disk at a given instant when its angular velocity about its axle, OD, is ω_1 and its angular velocity around the curve (also its angular velocity about an axis through D perpendicular to plane of paper) is Ω_1. Let A_2B_2 represent the disk after an interval of time dt, during which the angular velocities have increased to ω_2 and Ω_2. The plane of the disk during the time dt turns through an angle $d\theta$. Let H denote angular momentum. The angular momentum of the disk at the first instant is represented by $H_1 = (I\omega)_1$, and after the interval dt it is $H_2 = (I\omega)_2$. The change in H is ab (exaggerated), Fig. 222. This change is due to a change in direction of H, represented by ac, plus vectorially the change in magnitude represented by cb. A couple is always necessary to produce a change in H, the magnitude of the couple being the rate of change of H, the plane in which it acts being perpendicular to the vector

representing the rate of change of H and its sense of rotation being determined from the direction of the arrow.

The rate of change in the magnitude of H is

$$\frac{cb}{dt}=\frac{dH_1}{dt}=\frac{d(I\omega)_1}{dt}=I\frac{d\omega_1}{dt}=I\alpha_1,$$

I being constant.

The direction of cb is perpendicular to the plane of the disk. Hence a couple $C_1=I\alpha_1$ acts in the plane of the disk causing the disk to roll faster.

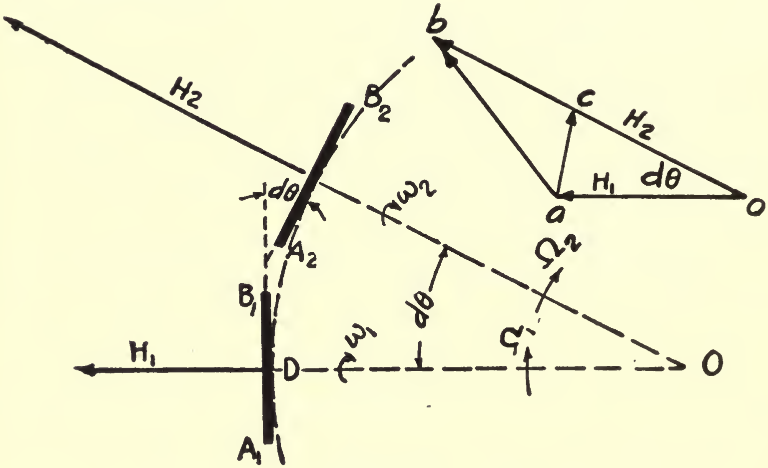


FIG. 222.

The rate of change of the direction of H is

$$\frac{ac}{dt}=\frac{H_1 d\theta}{dt}=I\omega_1\Omega,$$

cb and ac are perpendicular to each other, for the infinitesimal angle $d\theta$. Hence a couple $C_2=I\omega_1\Omega_1$ acts in the plane of the paper (perpendicular to plane of disk) in a clockwise direction. In other words, the disk would tend to overturn unless a couple C_2 acted to prevent the overturning.

As already shown $C_2=I\omega\Omega$ is the gyroscopic couple. If ω is constant $C_1=$zero, the total change in H then being ac and the only couple required to maintain the velocity Ω being the gyroscopic couple.

149. Precession.—When the body is allowed to precess, the external couple acting on the body will be resisted by the gyroscopic couple. This is readily seen by following the changes in the angular momentum in Fig. 220. When the couple Wl first acts, the change in the angular momentum produced is AB vertically upward. This requires a couple in the horizontal plane in an anti-clockwise direction, but since there are no bodies to develop or supply this couple, the disk turns (precesses) clockwise in the horizontal plane, developing the necessary couple ($I_y\dot{\psi}$) from the inertia of the disk. As soon as precession starts, however, the change AC is produced in the angular momentum. This requires a clockwise couple in the vertical plane to prevent the disk and axle from rotating anti-clockwise in the vertical plane. This couple is supplied by the two forces W having a moment Wl. Hence if the body is allowed to precess the external couple, Wl is resisted by the gyroscopic couple, or

$$Wl = I\omega\Omega = \frac{W}{g}k^2\omega\Omega,$$

from which the velocity of precession $\Omega = \frac{lg}{\omega k^2}$, k being the radius of gyration of the disk and axle with respect to the y axis.

If precession is prevented, a couple $C = I\omega\Omega$ must be set up in the horizontal plane, Ω here being he angular velocity at any instant produced by Wl, which will be the same whether the disk rotates or not, there being no resistance offered to the external couple unless precession is allowed. This explains why a heavy rotating flywheel or armature on board a ship, with axle horizontal and athwartship, will offer no more resistance to the rolling of the ship than if not rotating. The bearings of the axles, however, must exert a large couple $C = I\omega\Omega$ in a horizontal plane, which tends to " nose around " the ship. Ω here represents the angular velocity of roll. It is evident that the rotating bodies aboard ship will be subjected to less stress when their axes are parallel to the fore and aft direction, in which case Ω would be due to the pitching of the ship.

If an external couple is applied in a horizontal plane, Fig. 220, to increase or hurry the precession the disk and axle will rise, since $I\omega\Omega > Wl$. This is the principle employed in Brennen's

monorail car, the precession being hurried by the rolling of the axle of the revolving flywheels, extended, on a shelf attached to the side of the car. This principle is also used in the "active type" of gyroscope for stabilizing ships. In this case the precession is hurried by means of a precession engine, which acts after the ship has rolled a very small degree, thus producing a gyroscopic righting couple just sufficient to extinguish the roll. Since the roll is checked in its incipiency only a small amount of work is done. The stresses produced in the hull of the ship are also small for the same reason. The weight and displacement of the gyroscope may likewise be small.

As already pointed out, energy is dissipated in establishing precession. In order to start the disk and axle, Fig. 220, precessing the axle must dip due to the couple Wl through an angle $d\theta$. The work done during this displacement is $Wld\theta$. This work generates a certain amount of kinetic energy which appears as the kinetic energy of precession which is expressed by $\frac{1}{2}I_y\Omega^2$. The kinetic work $Wld\theta$, however, is not all expended in producing this energy, as may be shown as follows: When the axle dips an angle $d\theta$, the component of the angular momentum, $I\omega$, parallel to the y axis, is $I\omega \sin d\theta$. The total angular momentum with respect to the Y axis must be zero (conservation of angular momentum), hence the angular momentum of precession $I_y\Omega$ must neutralize this component, or

$$I\omega \sin d\theta = I_y\Omega,$$

multiplying by $\frac{1}{2}\Omega$ we get,

$$\tfrac{1}{2}I\omega\Omega \sin d\theta = I_y\Omega^2.$$

But

$$Wl = I\omega\Omega.$$

Therefore

$$\tfrac{1}{2}Wld\theta = \tfrac{1}{2}I_y\Omega^2,$$

that is, one-half of the work expended is transformed into kinetic energy of precession; the other half is dissipated through the friction which reduces the surging.

For those interested in the various ways in which gyroscopic motion may arise and in the progress which has been made in the application of the gyroscope to the stabilizing of ships, mono-

rail cars, and aeroplanes, to the guiding of torpedoes, to compasses, and to the Griffin grinding mill the following references are appended:

1. Journal of The Franklin Institute, May, 1913.
2. American Machinist, August 7 and 14, 1913.
3. Iron Age, December 1, 8, 15, and 22, 1910.
4. Scientific American Supplement, January 26, 1907.
5. Popular Science Monthly, July, 1909.

CHAPTER IX

CRITICAL SPEEDS AND VIBRATIONS

150. Introductory.—In the preceding chapters the links of all the mechanisms discussed have been regarded as rigid, and the distortions of the members under the action of the forces applied have been neglected. In some cases, notably in revolving shafts, there may be appreciable deflections; and sometimes vibrations are set up which may have serious consequences. At certain "critical speeds" the deflections and vibrations become extremely severe. In design of steam turbines and other high-speed machines care must be taken to see that the speeds employed are far removed from the critical values. The purpose of this chapter is to discuss in an elementary way the character and causes of such deflections and vibrations.

151. Revolving Shaft Loaded at Middle.—Let the shaft, Fig. 223, be supported at the ends and carry at the middle a disk or pulley of mass M. Let the center of gravity of this pulley be at a distance s from the center of the shaft. If the pulley revolves at an angular velocity ω the shaft will bend under the action of the centrifugal force. Let y be the deflection at the middle of the shaft. Then the distance of the center of gravity from the axis of rotation is

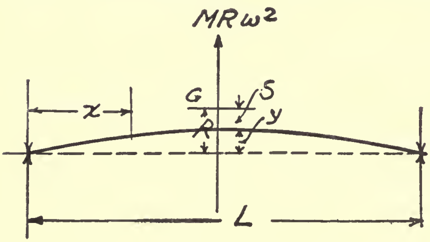


Fig. 223.

$$R = y + s. \qquad (1)$$

The centrifugal force exerted by the pulley is

$$C = MR\omega^2 = M(y+s)\omega^2. \qquad (2)$$

The centrifugal force is balanced by the elastic force tending to straighten the shaft. From the laws of mechanics of materials

$$C=\frac{48EIy}{L^3}, \qquad (3)$$

where E is the modulus of elasticity in tension (Young's modulus), I is the moment of inertia of the cross-section of the shaft, and L is the length of the shaft. Then

$$MR\omega^2=M(y+s)\omega^2=\frac{48EIy}{L^3}=Ma^2y, \qquad (4)$$

where

$$a^2=\frac{48EI}{ML^3}.$$

Therefore

$$y=\frac{s\omega^2}{a^2-\omega^2}. \qquad (5)$$

Evidently if $\omega=a$, y becomes infinite. The interpretation of this statement is that when $\omega=a$, the deflection will increase until stopped by the limitations of construction, or until the shaft receives a permanent set.

The speed $\omega=a$ is called the *critical speed* of the shaft. If ω becomes greater than a, the value of y becomes negative. In other words, the deflection y is opposite in direction to the eccentricity s. The position assumed by the shaft at speeds above the critical value is illustrated in Fig. 224.

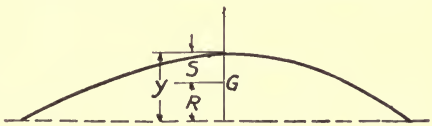


FIG. 224.

As ω increases above the critical value, y becomes less, until in the limit when ω becomes infinite, $y=-s$. That is to say, as the speed increases above the critical value the center of gravity moves inward toward the axis of rotation, reaching this axis when the speed becomes infinite.

The centrifugal force is given by the equation

$$C=M(y+s)\omega^2=\left(\frac{sa^2}{a^2-\omega^2}M\omega^2\right)=\frac{Ma^2s(\omega/a)^2}{1-(\omega/a)^2}. \qquad (6)$$

In Fig. 225 the values of $\frac{C}{Ma^2s}$ are plotted as ordinates against $\frac{\omega}{a}$ as abscissa. Fig. 225 shows that the centrifugal force increases rapidly up to the critical speed, and then decreases, approaching $-Ma^2s$ as a limit as ω approaches infinity.

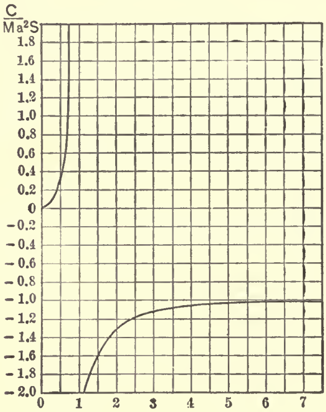


FIG. 225.

If the shaft were held in rigid bearings instead of being simply supported at the ends, the same argument holds good, the only change being in the value of a. In this case

$$a^2=\frac{384EI}{5ML^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

152. Natural Period of Vibration.—Let the shaft shown in Fig. 223 be kept from revolving and be bent by a force P acting at the middle. Then from the laws of mechanics of materials

$$P=\frac{48EIy}{L^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the shaft is now released it will spring back toward its original shape. The mass M will be given an acceleration

$$-\frac{d^2y}{dt^2}=\frac{P}{M}=\frac{48EIy}{ML^3}=a^2y, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where

$$a^2=\frac{48EI}{ML^3}.$$

The general solution of the differential Equation (2) is

$$y=A\sin at+B\cos at, \quad . \quad . \quad . \quad . \quad (3)$$

where A and B are arbitrary constants. These constants can be evaluated by putting

$$y=y_0, \text{ and } \frac{dy}{dt}=0 \text{ when } t=0.$$

From Equation (3)

$$\frac{dy}{dt}=a(A\cos at-B\sin at). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When

$$t=0,\ \sin\ at=0 \quad \text{and} \quad \frac{dy}{dt}=0.$$

Therefore

$$A=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

when

$$t=0, \quad y=B.$$

Therefore

$$B=y_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Equation (3) then reduces to

$$y=y_0\cos at. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Evidently $y=y_0$ whenever $\cos at=1$; that is when

$$at=2\pi,\ 4\pi,\ 6\pi,\ \text{etc.}$$

The time of a complete vibration is therefore

$$\tau=\frac{2\pi}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the number of complete vibrations per second is

$$\frac{1}{\tau}=\frac{a}{2\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

From Equation (6) of Art. 151, the number of complete revolutions per second when the shaft is running at its critical speed is

$$\frac{\omega}{2\pi}=\frac{a}{2\pi}.$$

Since a has the same meaning in Arts. 151 and 152, it follows that the critical speed measured in revolutions per second is the same as the natural speed of vibration of the shaft. This statement holds equally good when the shaft is held in rigid bearings instead of being simply supported at the ends.

153. Critical Speed of Uniformly Loaded Shaft.—Let the shaft in Fig. 226 carry a uniform load of w pounds mass per unit length, and let the shaft revolve at an angular velocity ω. Consider a section of the shaft of length dx and situated at a distance x from the left support. Let the mass wdx, of this section have its center of gravity at a distance s from the center of the shaft, and let the deflection of the shaft at this point be $y=f(x)$. Then the centrifugal force of the mass wdx is given by the equation

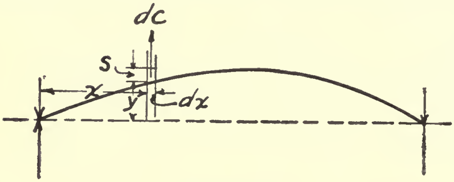


FIG. 226.

$$dC=(y+s)wdx\omega^2. \quad \quad (1)$$

From the principles of mechanics of materials the bending moment at the section under consideration is

$$M=EI\frac{d^2y}{dx^2}, \quad \quad (2)$$

and the shear is

$$V=\frac{dM}{dx}=EI\frac{d^3y}{dx^3}. \quad \quad (3)$$

The shear at a distance $x+dx$ from the left support is

$$V+dV=V+\frac{dV}{dx}dx=EI\frac{d^3y}{dx^3}+EI\frac{d^4y}{dx^4}dx. \quad \quad (4)$$

The difference between the shears at the sections x and $x+dx$ must balance the centrifugal force of the mass between these sections. Therefore

$$dV=EI\frac{d^4y}{dx^4}dx=dC=w\omega^2(y+s)dx, \quad \quad (5)$$

or

$$\frac{d^4y}{dx^4}=\frac{w\omega^2(y+s)}{EI}. \quad \quad (6)$$

Equation (6) is a linear differential equation of the fourth order. The complementary equation is

$$\frac{d^4y}{dx^4}-\frac{w\omega^2y}{EI}=0. \quad \quad (7)$$

The complementary function is therefore

$$y_0 = Ae^{kx} + Be^{-kx} + C \cos kx + D \sin kx, \quad . \quad . \quad . \quad (8)$$

where

$$k = \sqrt[4]{\frac{w\omega^2}{EI}}.$$

The general solution of Equation (6) is therefore

$$y = Ae^{kx} + Be^{-kx} + C \cos kx + D \sin kx + P(x), \quad . \quad (9)$$

where $P(x)$ is a particular solution of Equation (6).

The arbitrary constants A, B, C and D are found from the conditions at the ends of the shaft. If the shaft is simply supported at the ends, $y=0$ and $\frac{d^2y}{dx^2}=0$ when $x=0$ and when $x=L$. Purring these values in Equation (9) we get

$$A+B+C+P(O)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$A+B-C+\frac{1}{k^2}\frac{d^2P(O)}{dx^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$Ae^{kL}+Be^{-kL}+C \cos kL+D \sin kL+P(L)=0, \quad . \quad (12)$$

$$Ae^{kL}+Be^{-kL}-C \cos kL-D \sin kL+\frac{1}{k^2}\frac{d^2P(L)}{dx^2}=0 \quad . \quad (13)$$

Solving Equations (10), (11), (12) and (13)

$$A = \frac{P(O)e^{-kL}-P(L)+\frac{1}{k^2}\left[\frac{d^2P(O)}{dx^2}e^{-kL}-\frac{d^2P(L)}{dx^2}\right]}{2(e^{kL}-e^{-kL})}. \quad . \quad . \quad . \quad (14)$$

$$B = \frac{P(O)e^{kL}-P(L)+\frac{1}{k^2}\left[\frac{d^2P(O)}{dx^2}e^{kL}-\frac{d^2P(L)}{dx^2}\right]}{-2(e^{kL}-e^{-kL})}. \quad . \quad . \quad . \quad . \quad (15)$$

$$C = -\tfrac{1}{2}\left[P(O)-\frac{1}{k^2}\frac{d^2P(O)}{dx^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$D = \frac{P(O) \cos kL-P(L)+\frac{1}{k^2}\left[\frac{d^2P(L)}{dx^2}-\frac{d^2P(O)}{dx^2} \cos kL\right]}{2 \sin kL}. \quad . \quad (17)$$

From Equation (9) it follows that y will become infinite if any one of the arbitrary constants A, B, C, D becomes infinite. From Equations (14) to (17) it is readily seen that the only one of these

constants which can have an infinite value is D. If $\sin kL=0$, D and consequently y increases without limit, unless the numerator of the right-hand member of Equation (17) vanishes when $\sin kL=0$. The critical speeds are therefore such that $\sin kL=0$, or

$$kL=\pi,\ 2\pi,\ 3\pi,\ \ldots \qquad (18)$$

Substituting the value of k in Equation (18)

$$L\sqrt[4]{\frac{w\omega^2}{EI}}=\pi,\ 2\pi,\ 3\pi,\ \text{etc.},$$

or

$$\omega=\frac{\pi 2}{L^2}\sqrt{\frac{EI}{w}},\quad \frac{4\pi^2}{L^2}\sqrt{\frac{EI}{w}},\quad \frac{9\pi^2}{L^2}\sqrt{\frac{EI}{w}},\ \text{etc.} \qquad (19)$$

For a uniformly loaded shaft there is therefore a series of critical speeds, these speeds being proportional to the squares of the natural numbers. That is

$$\omega_1 : \omega_2 : \omega_3 : \omega_4,\ \text{etc.}=1^2 : 2^2 : 3^2 : 4^2,\ \text{etc.} \qquad (20)$$

The investigation of the exceptional cases where D is not infinite even when $\sin kL$ vanishes, requires a knowledge of the form of the particular integral $P(x)$, which in turn depends on the character of the function s. Since s is a perfectly arbitrary function of x, the numerator of the right-hand member of Equation (17) is, in general, not equal to zero. It is easily possible, however, to find cases where this expression vanishes when $\sin kL=0$, and where therefore D may not become infinite even if $\sin kL$ may be zero. For example let the distance s, Fig. 226, be a constant $s=a$. Then Equation (6) may be written

$$\frac{d^4(y+a)}{dx^4}=k^4(y+a).$$

The general solution of this equation is

$$y+a=Ae^{kx}+Be^{-kx}+C\cos kx+D\sin kx.$$

It follows that the particular solution reduces to

$$P(x)=-a$$

Substituting this value in Equation (17)

$$D=\frac{a(1-\cos kL)}{\sin kL}.$$

When $\sin kL = 0$,

$\cos kL = \pm 1.$

If $\cos kL = +1,\ D = 0.$

If $\cos kL = -1,\ D = \infty.$

Hence the critical speeds will be such that $\cos kL = -1$, or

$$kL = \pi,\ 3\pi,\ 5\pi,\ 7\pi,\ 9\pi, \text{ etc.}$$

Therefore

$$\omega = \frac{\pi^2}{L^2}\sqrt{\frac{EI}{w}},\quad \frac{9\pi^2}{L^2}\sqrt{\frac{EI}{w}},\quad \frac{25\pi^2}{L^2}\sqrt{\frac{EI}{w}} \text{ etc.}$$

The disappearance of the critical speeds corresponding to the values $kL = 2\pi$, 4π, 6π, etc., may be explained as follows. The deflection of the shaft as given by Equation (9) is made up of a series of terms such as shown in Fig. 227. In the case of a symmetrically loaded shaft the deflections shown in Fig. 227 (*b*), (*d*), (*f*) etc., cannot take place. There remain therefore only the deflections shown in Fig. 227 (*a*), (*c*), (*e*), etc. The critical speeds correspond to the conditions when any one of the deflections increases without limit. When the shaft is so loaded that any of the deflections vanishes at all speeds the corresponding critical speed also vanishes.[1]

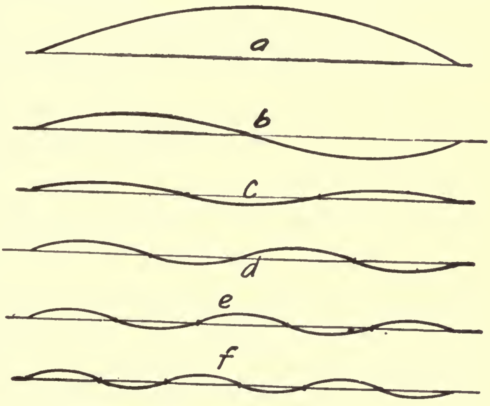


FIG. 227.

[1] The condition under which one or more of the critical speeds will disappear is given by the equation,

$$P(x) - \frac{1}{k^2}\frac{d^2P(x)}{dx^2} \equiv k\left[\cos kx \int \sin kx\, s\, dx - \sin kx \int \cos kx\, s\, dx\right].$$

In the example given above $s = a$, and $P(x) = -a$. Substituting these values for s and $P(x)$,

$$k\left[-\frac{1}{k} a \cos^2 kx - \frac{1}{k} a \sin^2 kx\right] = -a.$$

For the development of this criterion the writers are indebted to Dr. Cyril A. Nelson of the Department of Mathematics, Western Reserve University.

154. Vibrations of Uniformly Loaded Shaft.—Let the shaft in Fig. 226 be distorted by any set of external forces and then released. There will be set up a series of vibrations of varying amplitudes and periods according to the manner in which the shaft was originally bent. Let y be the deflection of any point of the shaft at a distance x from the left support, and at the time t. Then

$$y=f(x, t).$$

At a distance x from the left support take a section of the shaft whose length is dx. The mass of this section is wdx, or

$$dM=wdx. \quad \ldots \ldots \ldots \quad (1)$$

As in the preceding article the shear on the section is

$$V=EI\frac{\delta^3 y}{\delta x^3},$$

and the net force on the element under consideration is

$$dV=EI\frac{\delta^4 y}{\delta x^4}dx. \quad \ldots \ldots \ldots \quad (2)$$

(The partial derivative is used to indicate that the time is considered constant in this equation). Then the acceleration of the mass wdx is

$$\frac{EI}{w}\frac{\delta^4 y}{\delta x^4}=-\frac{\delta^2 y}{\delta t^2}. \quad \ldots \ldots \ldots \quad (3)$$

In order to find the solution of Equation (3) assume

$$\frac{\delta^4 y}{\delta x^4}=m^4 y, \quad \ldots \ldots \ldots \quad (4)$$

and

$$\frac{\delta^2 y}{\delta t^2}=-n^2 y. \quad \ldots \ldots \ldots \quad (5)$$

Therefore

$$\frac{EI}{w}m^4=n^2, \quad \ldots \ldots \ldots \quad (6)$$

and

$$n=m^2\sqrt{\frac{EI}{w}}. \quad \ldots \ldots \ldots \quad (7)$$

An expression of the form

$$y=(Ae^{mx}+Be^{-mx}+C\cos mx+D\sin mx)(E\cos nt+F\sin nt). \quad (8)$$

satisfies these conditions.[1]

At the ends of the shaft there is no deflection and no bending moment. Therefore when $x=0$, $y=0$ and $\frac{\delta^2 y}{\delta x^2}=0$. Substituting these values in Equation (8) and reducing, it follows that

$$A+B+C=0, \quad \ldots \quad (9)$$

and

$$A+B-C=0, \quad \ldots \quad (10)$$

and therefore

$$C=0, \quad \ldots \quad (11)$$

and

$$A=-B. \quad \ldots \quad (12)$$

If the time is measured from the instant of release it follows that when

$$t=0, \frac{\delta y}{\delta t}=0.$$

In other words, at the moment of release the particles of the shaft have no velocity. Substituting the values of B and C in Equation (8) and differentiating with respect to t,

$$\frac{\delta y}{\delta t}=n(Ae^{mx}-Ae^{-mx}+D\sin mx)(-E\sin nt+F\cos nt)=0. \quad (13)$$

when $t=0$. As $\sin nt=0$ if $t=0$ it follows that

$$F=0. \quad \ldots \quad (14)$$

Equation (8) is now reduced to

$$y=(Ae^{mx}-Ae^{-mx}+D\sin mx)E\cos nt. \quad (15)$$

[1] The general solution of Equation (3) may be written in the form of Equation (8), substituting for A, B, C and D four arbitrary functions of t, and for E and F arbitrary functions of x. The validity of this solution may easily be shown by forming the derivatives $\frac{\delta^4 y}{\delta x^4}$ and $\frac{\delta^2 y}{\delta y^2}$ and substituting the values thus found in Equation (3). The argument which follows can be adapted with slight modifications to the case where arbitrary functions are used instead of arbitrary constants. For the sake of simplicity A, B, C, D, E, and F are treated as constants in the text. The results expressed in Equations (26), (28) and (29) are identical with those obtained using the general solution.

When $x=L$, $y=0$ and $\frac{\delta^2 y}{\delta x^2}=0$. Substituting these values in Equation (15) it follows that

$$A(e^{mL}-e^{-mL})+D \sin mL=0, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$A(e^{mL}-e^{-mL})-D \sin mL=0. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Hence

$$A=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$$D \sin mL=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From Equations (15) and (18) it is evident that if $D=0$, $y=0$ and there is no deflection. As this is contrary to the hypothesis that the shaft is bent, it follows from Equation (19) that

$$\sin mL=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Therefore

$$mL=\pi,\ 2\pi,\ 3\pi,\ 4\pi, \text{ etc}, \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and

$$n=m^2\sqrt{\frac{EI}{w}}=\frac{\pi^2}{L^2}\sqrt{\frac{EI}{w}},\quad \frac{4\pi^2}{L^2}\sqrt{\frac{EI}{w}},\quad \frac{9\pi^2}{L^2}\sqrt{\frac{EI}{w}}, \text{ etc.} \quad . \quad (22)$$

Equation (8) is now finally reduced to

$$y=c \sin mx \cos nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $c=DE.$

It has been shown that m and n can have a series of values m_1, m_2, m_3, etc., and n_1, n_2, n_3, etc., which are defined by Equations (21) and (22). Any pair of these values satisfies Equation (23). Furthermore the sum of the series of terms

$c_1 \sin m_1x \cos n_1t+c_2 \sin m_2x \cos n_2t+c_3 \sin m_3x \cos n_3t+$, etc.,

is also a solution of Equation (23). We can therefore write

$$y=\sum_1^\infty c_s \sin m_sx \cos n_st, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where s can take all integral values from 1 to ∞.

At the time $t=0$ the shaft has been distorted by external forces into any desired form. Let the original deflections be represented by the equation

$$y=F(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Then when $t=0$,

$$y=F(x)=c_1\sin\frac{\pi x}{L}+c_2\sin\frac{2\pi x}{L}+c_3\sin\frac{3\pi x}{L}\quad . \quad . \quad . \quad (26)$$

This is a Fourier series. The coefficients c_1, c_2, c_3, etc., are determined by the equation

$$c_s=\frac{1}{L}\int_{-L}^{L}F(x)\sin\frac{s\pi x}{L}\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The vibrations of the shaft are therefore made up of a set of harmonic oscillations of different amplitudes and different periods. The amplitudes are given by the values of the constants c_1, c_2, etc., and the periods are given by the equations

$$\left.\begin{aligned} n_1t_1&=2\pi \quad\text{or}\quad t_1=\frac{2L^2}{\pi}\sqrt{\frac{w}{EI}}\\ n_2t_2&=2\pi \quad\text{or}\quad t_2=\frac{2L^2}{4\pi}\sqrt{\frac{w}{EI}}\\ n_3t_3&=2\pi \quad\text{or}\quad t_3=\frac{2L^2}{9\pi}\sqrt{\frac{w}{EI}}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (28)$$

etc. The number of vibrations per second are the reciprocals of t_1, t_2, etc.

$$\left.\begin{aligned} N_1&=\frac{\pi}{2L^2}\sqrt{\frac{EI}{w}}\\ N_2&=\frac{4\pi}{2L^2}\sqrt{\frac{EI}{w}}\\ N_3&=\frac{9\pi}{2L^2}\sqrt{\frac{EI}{w}}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

etc. From Equation (19) Art. 153 the critical speeds of such a shaft are found to be:

$$\omega_1=\frac{\pi^2}{L^2}\sqrt{\frac{EI}{w}}\text{ radians per second}=\frac{\pi}{2L^2}\sqrt{\frac{EI}{w}}\text{ revolutions per second.}$$

$$\omega_2=\frac{4\pi^2}{L^2}\sqrt{\frac{EI}{w}}\text{ radians per second}=\frac{4\pi}{2L^2}\sqrt{\frac{EI}{w}}\text{ revolutions per second.}$$

etc. Therefore the critical speeds expressed in revolutions per second are exactly the same as the speeds of the vibrations set up in the shaft.

155. Shaft Rotating in Fixed Bearings. Uniformly Loaded.—For the type of shaft illustrated in Fig. 228 Equations (1) to (9) of Art. 153 hold good. The difference in the solution consists simply in the determination of the arbitrary constants A, B, C, D of Equation (9), Art. 153. When the shaft rotates in fixed bearings $y=0$ and $\frac{dy}{dx}=0$ at the two ends, that is, where $x=0$ and where $x=L$.

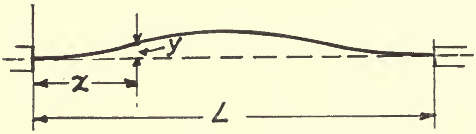


FIG. 228.

From Equation (9), Art. 153,

$$y=Ae^{kx}+Be^{-kx}+C\cos kx+D\sin kx+P(x). \quad . \quad (1)$$

Differentiating,

$$\frac{dy}{dx}=k(Ae^{kx}-Be^{-kx}-C\sin kx+D\cos kx)+\frac{dP(x)}{dx}. \quad (2)$$

Putting y and $\frac{dy}{dx}=0$ at the ends,

$$A+B+C+P(O)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$A-B+D+\frac{1}{k}\frac{dP(O)}{dx}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$Ae^{kL}+Be^{-kL}+C\cos kL+D\sin kL+P(L)=0. \quad . \quad . \quad (5)$$

$$Ae^{kL}-Be^{-kL}-C\sin kL+D\cos kL+\frac{1}{k}\frac{dP(L)}{dx}=0. \quad . \quad (6)$$

Solving these equations:

$$A=\frac{cg-bh}{bf-ag}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$B=\frac{ah-cf}{bf-ag}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$C=\frac{(b-a)h+(f-g)c}{bf-ag}-P(O), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$D=\frac{(b+a)h-(f+g)c}{bf-ag}-\frac{1}{k}\frac{dP(O)}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where
$$a=e^{kL}-\cos kL-\sin kL;$$
$$b=e^{-kL}-\cos kL+\sin kL;$$
$$c=P(O)\cos kL+P(L)-\frac{1}{k}\frac{dP(O)}{dx}\sin kL;$$
$$f=e^{kL}+\sin kL-\cos kL;$$
$$g=-e^{-kL}+\sin kL+\cos kL;$$
$$h=-\frac{1}{k}\frac{dP(O)}{dx}\cos kL+P(O)\sin kL+\frac{1}{k}\frac{dP(L)}{dx}.$$

If any of the arbitrary constants A, B, C, D are to become infinite, it must be when $bf-ag=0$.

Substituting the values of a, b, f, g, and expanding we get, after suitable reduction:

$$4[1-\tfrac{1}{2}(e^{kL}+e^{-kL})\cos kL]=4(1-\cos kL\cosh kL)=0. \quad (11)$$

Hence

$$\cos kL\cosh kL=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

This gives for the critical speeds the values corresponding to

$$kL=0,\ 4.729,\ \frac{5\pi}{2}-\Delta_1,\ \frac{7\pi}{2}+\Delta_2,\ \frac{9\pi}{2}-\Delta_3,\ \ldots \quad . \quad . \quad (13)$$

where Δ_1, Δ_2, Δ_3, . . . are very small quantities. The critical speeds are such, therefore, that kL is slightly greater than $\frac{3\pi}{2}$, $\frac{7\pi}{2}$, $\frac{11\pi}{2}$, $\frac{15\pi}{2}$, . . . , or slightly less than $\frac{5\pi}{2}$, $\frac{9\pi}{2}$, $\frac{13\pi}{2}$, ... Since $k=\sqrt[4]{\frac{w\omega^2}{EI}}$, the critical speeds are approximately given by

$$\omega=\frac{9\pi^2}{4L^2}\sqrt{\frac{EI}{w}},\ \frac{25\pi^2}{4L^2}\sqrt{\frac{EI}{w}},\ \frac{49\pi^2}{4L^2}\sqrt{\frac{EI}{w}},\ \ldots .$$

156. Vibrations of Uniformly Loaded Shaft Fixed at Both Ends.—Let the shaft represented in Fig. 228 be bent into any desired form by external forces, and then released. A set of vibrations will be set up which follow the laws derived in Equations (1) to (8) of Art. 154. That is

$$\frac{EI}{w}\frac{\delta^4 y}{\delta x^4}+\frac{\delta^2 y}{\delta t^2}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y=(Ae^{mx}+Be^{-mx}+C\cos mx+D\sin mx)(E\cos nt+F\sin nt)\ , \quad (2)$$

$$\frac{\delta^4 y}{\delta x^4}=m^4 y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{\delta^2 y}{\delta t^2}=-n^2 y, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$n^2=\frac{EI}{w}m^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The end conditions of the shaft give:

$y=0$ when $x=0$, and when $x=L$ and

$\frac{\delta y}{\delta x}=0$ when $x=0$, and when $x=L$.

If the time is measured from the instant of release,

$\frac{\delta y}{\delta t}=0$ when $t=0$.

Therefore $F=0$, exactly as in Art. 154.

When $x=0$,

$$y=(A+B+C)E\cos nt=0,$$

and therefore

$$A+B+C=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also

$$\frac{\delta y}{dx}=m(A-B+D)E\cos nt=0,$$

and therefore

$$A-B+D=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When $x=L$,

$$y=(Ae^{mL}+Be^{-mL}+C\cos mL+D\sin mL)E\cos nt=0,$$

and therefore

$$Ae^{mL}+Be^{-mL}+C\cos mL+D\sin mL=0. \quad . \quad . \quad (8)$$

Also

$$\frac{\delta y}{\delta x}=m(Ae^{mL}-Be^{-mL}-C\sin mL+D\cos mL)E\cos nt=0,$$

whence

$$Ae^{mL}-Be^{-mL}-C\sin mL+D\cos mL=0. \quad . \quad . \quad (9)$$

Substituting in Equations (8) and (9) the values of C and D from Equations (6) and (7)

$$A(e^{mL}-\cos mL-\sin mL)+B(e^{-mL}-\cos mL+\sin mL)=0, \quad (10)$$

$$A(e^{mL}+\sin mL-\cos mL)+B(-e^{-mL}+\cos mL+\sin mL)=0. \quad (11)$$

Equations (10) and (11) can be written in the form

$$aA+bB=0, \quad (12)$$

$$cA+dB=0. \quad (13)$$

Eliminating B from Equations (12) and (13)

$$A(bc-ad)=0. \quad (14)$$

Therefore $A=0$, or

$$bc-ad=0. \quad (15)$$

Eliminating A from Equations (12) and (13)

$$B(bc-ad)=0, \quad (16)$$

and therefore $B=0$, or

$$bc-ad=0. \quad (17)$$

If both A and B are zero, C and D must also be zero, and there is no deflection of the shaft. This is contrary to hypothesis, and it is therefore necessary to adopt the other solution, namely

$$bc-ad-0. \quad (18)$$

Substituting in Equation (18) the values of a, b, c and d from Equations (10) and (11), and reducing,

$$4[1-\tfrac{1}{2}\cos mL(e^{mL}+e^{-mL})]=4(1-\cos mL\cosh mL)=0. \quad (19)$$

Therefore

$$\cos mL\cosh mL=1, \quad (20)$$

and mL has exactly the same values as those found for kL in the preceding article, that is

$$mL=0,\ \frac{3\pi}{2}-\Delta_1,\ \frac{5\pi}{2}+\Delta_2,\ \frac{7\pi}{2}-\Delta_3, \text{ etc.},$$

where Δ_1, Δ_2, Δ_3, etc., are small quantities. The value $mL=0$ is inadmissable and may be discarded. The remaining values of mL are approximately

$$mL=\frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2}, \text{ etc.} \quad (21)$$

Hence

$$m=\frac{3\pi}{2L}, \frac{5\pi}{2L}, \frac{7\pi}{2L}, \text{ etc.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$n=m^2\sqrt{\frac{EI}{w}}=\frac{9\pi^2}{4L^2}\sqrt{\frac{EI}{w}}, \frac{25\pi^2}{4L^2}\sqrt{\frac{EI}{w}}, \frac{49\pi^2}{4L^2}\sqrt{\frac{EI}{w}}, \text{ etc.} \quad (23)$$

The period of oscillation, τ, is given by:

$$n\tau=2\pi,$$

so that

$$\tau=\frac{2\pi}{n}=\frac{8L^2}{9\pi}\sqrt{\frac{w}{EI}}, \frac{8L^2}{25\pi}\sqrt{\frac{w}{EI}}, \frac{8L^2}{49\pi}\sqrt{\frac{w}{EI}} \dots . \quad . \quad . \quad (24)$$

Oscillations per second

$$=\frac{1}{\tau}=\frac{9\pi}{8L^2}\sqrt{\frac{EI}{w}}, \frac{25\pi}{8L^2}\sqrt{\frac{EI}{w}}, \frac{49\pi}{8L^2}\sqrt{\frac{EI}{w}} \dots \quad . \quad . \quad . \quad (25)$$

From Art. 155 the critical speeds are given by:

$$\omega=\frac{9\pi^2}{4L^2}\sqrt{\frac{EI}{w}}, \frac{25\pi^2}{4L^2}\sqrt{\frac{EI}{w}}, \frac{49\pi^2}{4L^2}\sqrt{\frac{EI}{w}} \dots \text{ radians per sec.}$$

$$=\frac{9\pi}{8L^2}\sqrt{\frac{EI}{w}}, \frac{25\pi}{8L^2}\sqrt{\frac{EI}{w}}, \frac{49\pi}{8L^2}\sqrt{\frac{EI}{w}} \dots \text{ rev. per second.}$$

In other words, the critical speeds, expressed in revolutions per second are equal to the natural periods of vibration of the shaft.

The constants A, B, C, D, E. are determined in the same manner as in Art. 154.

157. Inclination of Rotating Disk.—In the preceding articles the unbalanced masses have been considered as concentrated at a point, or as concentrated along a line at a known distance s from the center line of the shaft. In other words, there is considered to be only one mass in any plane drawn normal to the center line of the shaft, and this mass is regarded as concentrated at a point. In practical constructions, such as the shaft and wheel of a turbine, the load in any transverse plane is distributed over the area of the wheel or disk. This causes a modification of the laws of critical velocities and of vibrations of the shaft, on account of the angle through which the disk is turned when the shaft is bent.

As an example consider the shaft shown in Fig. 229. The shaft is supported in one bearing and carries a disk at its free end. If the mass is regarded as concentrated at its center of gravity G, the critical speed of the shaft is readily shown to be:

$$\omega=\sqrt{\frac{3EI}{ML^3}}.$$

If, however, the mass is distributed over the area of the disk a different value of the critical speed will be found.

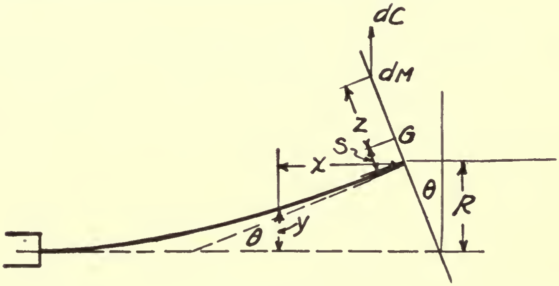


FIG. 229.

Let s = distance of G from center line of the shaft;
θ = angle of inclination of shaft at free end;
dM = mass of any element of the disk;
z = distance from G to dM, measured at right angles to the center line of the shaft;
R = deflection at free end of shaft.

Then, for small values of θ the following relations hold:
Distance of dM from center of rotation $=R+s+z$.
Centrifugal force of

$$dM=dC=(R+s+z)\omega^2dM. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Moment of dC about $G=dT_0=(R+s+z)\omega^2dMz\sin\theta$ . . . (2)

And since for small angles, $\sin\theta=\theta$,

$$dT_0=(R+s+z)\theta\omega^2zdM.$$

Moment of centrifugal force about $G=T_0=\int dT_0$

$$=\omega^2\theta\int(R+s+z)zdM=\omega^2\theta\left[(R+s)\int zdM+\int z^2dM\right]. \quad . \quad (3)$$

From the definition of the center of gravity:

$$\int z dM = 0.$$

Hence,

$$T_0 = \omega^2 \theta \int z^2 dM = \omega^2 \theta J. \quad \ldots \ldots \quad (4)$$

where J is the moment of inertia of the disk with respect to an axis passing through G and normal to the center line of the shaft. There is therefore a couple of moment $\omega^2\theta J$ which is caused by the inclination of the disk, and which tends to straighten the shaft.

The centrifugal force $C = M(R+s)\omega^2$, tends to bend the shaft. The moment of the centrifugal force about any section at a distance x from the free end is:

$$T_1 = M(R+s)\omega^2 x.$$

The net bending moment is therefore:

$$T = T_1 - T_0 = M(R+s)\omega^2 x - \omega^2 \theta J. \quad \ldots \ldots \quad (5)$$

From the laws of mechanics of materials

$$T = EI\frac{d^2y}{dx^2} = M(R+s)\omega^2 x - \omega^2\theta J.$$

Integrating twice:

$$EI\frac{dy}{dx} = \frac{M(R+s)\omega^2 x^2}{2} - \omega^2\theta J x + C_1, \quad \ldots \ldots \quad (6)$$

$$EIy = \frac{M(R+s)\omega^2 x^3}{6} - \frac{\omega^2\theta J x^2}{2} + C_1 x + C_2. \quad \ldots \quad (7)$$

When

$$x = L,\ y = 0 \text{ and } \frac{dy}{dx} = 0.$$

Hence

$$C_1 = \omega^2\theta J L - \frac{M(R+s)\omega^2 L^2}{2}, \quad \ldots \ldots \quad (8)$$

$$C_2 = -\frac{\omega^2\theta J L^2}{2} + \frac{M(R+s)\omega^2 L^3}{3}. \quad \ldots \ldots \quad (9)$$

When $x = 0$, $y = R$, and $\frac{dy}{dx} = -\tan\theta = -\theta$ for small deflections.

Then from Equation (6)

$$-EI\theta=C_1=\omega^2\theta JL-\frac{M(R+s)\omega^2L^2}{2}. \qquad (10)$$

Hence

$$\theta=\frac{M(R+s)L^2\omega^2}{2(EI+\omega^2JL)}. \qquad (11)$$

When $x=0$, $y=R$. Then from Equation (7)

$$EIR=C_2=-\frac{\omega^2\theta JL^2}{2}+\frac{M(R+s)\omega^2L^3}{3}$$

$$=-\frac{\omega^2JL^2M(R+s)\omega^2L^2}{4(EI+\omega^2JL)}+\frac{M(R+s)\omega^2L^3}{3}$$

$$=(R+s)\left[-\frac{J\omega^4ML^4}{4(EI+\omega^2JL)}+\frac{\omega^2ML^3}{3}\right]. \qquad (12)$$

Therefore

$$R=\frac{sM\omega^2L^3}{3EI}\left\{\frac{1-\dfrac{3}{4\left(\dfrac{EI}{J\omega^2L}+1\right)}}{1-\dfrac{M\omega^2L^3}{3EI}\left[1-\dfrac{3}{4\left(\dfrac{EI}{J\omega^2L}+1\right)}\right]}\right\}. \qquad (13)$$

Let

$$1-\frac{3}{4\left(\dfrac{EI}{J\omega^2L}+1\right)}=Z.$$

And let

$$1-\frac{M\omega^2L^3}{3EI}Z=N.$$

Then Equation (13) reduces to:

$$R=\frac{Ms\omega^2L^3Z}{3EIN},[1] \qquad (14)$$

R becomes infinite when $N=0$. The critical speed is therefore such that N vanishes. Putting $N=0$ and clearing of fractions:

$$ML^4J\omega^4+(4EIML^3-12\,EIJL)\omega^2-12\,E^2I^2=0. \qquad (15)$$

Hence,

$$\omega^2=2\,EI\left[\frac{-ML^2+3J+\sqrt{M^2L^4-3JML^2+9J^2}}{ML^3J}\right]. \qquad (16)$$

[1] Stodola: The Steam Turbine.

Example.—A steel shaft 2 inches in diameter overhangs its bearing by 30 inches and carries at its free end a steel disk $\frac{1}{2}$ inch thick and 30 inches in diameter. Neglecting the mass of the shaft find the critical speed of rotation.

Weight of disk $=\pi\times 15^2\times\frac{1}{2}\times 0.24=84.8$ pounds.

$$M=\frac{W}{g}=0.219,$$

g being expressed in inches per sec. per sec.

$$J=\frac{MD^2}{16}=\frac{0.219\times 900}{16}=12.32,$$

$$I=\frac{\pi d^4}{64}=0.7854,$$

E for steel $=30{,}000{,}000$ pounds per square inch.

Substituting these values in Equation (16)

$$\omega^2=13258,$$

$$\omega=115.1 \text{ radians per second}=18.34 \text{ rev. per sec.}$$

If the weight of 84.8 pounds were concentrated at the end of the shaft the critical speed would be given by

$$\omega=\sqrt{\frac{3EI}{ML^3}}=\sqrt{11954}=109.3 \text{ radians per second.}$$
$$=17.41 \text{ revolutions per second.}$$

The critical speed is thus seen to be increased due to the inclination of the disk.

158. Vibrations of Shaft Loaded with Disk at Free End.—Suppose the shaft shown in Fig. 229 to be held from rotating and to be bent by some external force. If the shaft is now released vibrations will be set up. It is required to find the speed of the vibrations of a shaft carrying a *disk* at the free end, as compared with the vibrations set up in a similar shaft carrying a *concentrated* load at the free end.

When the shaft is set in vibration the disk will have an angular acceleration $\frac{d^2\theta}{dt^2}$, and a linear acceleration $\frac{d^2R}{dt^2}$. To give the mass its linear acceleration a force is required:

$$F=M\frac{d^2R}{dt^2}.$$

The moment of this force about any section of the shaft at a distance x from the free end is:

$$T_1 = Mx\frac{d^2R}{dt^2}. \qquad (1)$$

The moment necessary to give the disk its angular acceleration is:

$$T_0 = J\frac{d^2\theta}{dt^2}. \qquad (2)$$

The total moment tending to bend the shaft is therefore:

$$T_1 + T_0 = EI\frac{\delta^2 y}{\delta x^2} = Mx\frac{d^2R}{dt^2} + J\frac{d^2\theta}{dt^2}. \qquad (3)$$

Integrating twice:

$$EI\frac{\delta y}{\delta x} = J\frac{d^2\theta}{dt^2}x + \frac{Mx^2}{2}\frac{d^2R}{dt^2} + C_1, \qquad (4)$$

$$EIy = J\frac{d^2\theta}{dt^2}\frac{x^2}{2} + M\frac{d^2R}{dt^2}\frac{x^3}{6} + C_1x + C_2, \qquad (5)$$

where C_1 and C_2 are functions of t.

When $x = L$, $y = 0$ and $\frac{\delta y}{\delta x} = 0$.

Therefore,

$$C_1 = -JL\frac{d^2\theta}{dt^2} - \frac{ML^2}{2}\frac{d^2R}{dt^2}, \qquad (6)$$

$$C_2 = \frac{JL^2}{2}\frac{d^2\theta}{dt^2} + \frac{ML^3}{3}\frac{d^2R}{dt^2}. \qquad (7)$$

When $x = 0$, $y = R$ and $\frac{\delta y}{\delta x} = -\tan\theta = -\theta$ for small deflections. Substituting these values in Equations (4) and (5):

$$-\theta = -\frac{JL}{EI}\frac{d^2\theta}{dt^2} - \frac{ML^2}{2EI}\frac{d^2R}{dt^2}. \qquad (8)$$

$$R = \frac{JL^2}{2EI}\frac{d^2\theta}{dt^2} + \frac{ML^3}{3EI}\frac{d^2R}{dt^2}. \qquad (9)$$

Eliminating $\frac{d^2\theta}{dt^2}$ between Equations (8) and (9)

$$R - \frac{L\theta}{2} = \frac{ML^3}{12EI}\frac{d^2R}{dt^2}, \qquad (10)$$

and therefore,

$$\theta=-\frac{2}{L}\left(\frac{ML^3}{12EI}\frac{d^2R}{dt^2}-R\right). \quad . \quad . \quad . \quad (11)$$

Eliminating $\frac{d^2R}{dt^2}$ from Equations (8) and (9)

$$R-\frac{2L\theta}{3}=-\frac{JL^2}{6EI}\frac{d^2\theta}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From Equation (11) differentiating twice,

$$\frac{d^2\theta}{dt^2}=-\frac{2}{L}\left(\frac{ML^3}{12EI}\frac{d^4R}{dt^4}-\frac{d^2R}{dt^2}\right). \quad . \quad . \quad . \quad (13)$$

Substituting in Equation (12) the values of θ and $\frac{d^2\theta}{dt^2}$ from Equations (11) and (13) and collecting terms

$$\frac{MJL^4}{12E^2I^2}\frac{d^4R}{dt^4}+\frac{3JL+ML^3}{3EI}\frac{d^2R}{dt^2}-R=0. \quad . \quad . \quad . \quad (14)$$

This is a linear differential equation of the fourth order. The auxiliary equation is

$$\frac{MJL^4}{12E^2I^2}m^4+\frac{3JL+ML^3}{3EI}m^2-1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The four roots of Equation (15) are given by

$$m=\pm\sqrt{\frac{2EI}{MJL^3}[3J+ML^2\pm\sqrt{9J^2+9JML^2+M^2L^4}]}. \quad . \quad (16)$$

Two of these roots are real and two are imaginary. Let

$$m_3=-m_4=k\sqrt{-1}$$

be the imaginary roots. Then the general solution of Equation (16) is

$$R=Ae^{m_1t}+Be^{-m_1t}+C\cos kt+D\sin kt. \quad . \quad . \quad . \quad (17)$$

The period of the vibrations is given by the value of k. If τ represents the period of vibration then

$$\tau k=2\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The number of vibrations per second is

$$\frac{1}{\tau}=\frac{k}{2\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Example.—Find the number of vibrations per second of the shaft described in the example of the preceding article.

As before

$$M = 0.219;$$

$$J = 12.32;$$

$$I = 0.7854;$$

$$E = 30,000,000.$$

Substituting these values in Equation (16)

$$k = 98.77.$$

$$\text{Vibrations per second} = \frac{98.77}{2\pi} = 15.72.$$

If the load were concentrated in a point at the free end, the vibrations per second would be 17.41, corresponding to the critical speed under these conditions. It follows that when the obliquity of the disk is taken into account, the critical speed is no longer the same as the natural speed of vibration of the shaft. The effect of the obliquity of the disk is to increase the critical speed and to reduce the rapidity of elastic vibration.

159. Other Systems of Loading.—In the preceding articles the only cases considered have been those of a single concentrated load or a load uniformly distributed along the shaft. Many other arrangements might be investigated along similar lines. Some of the arrangements which may occur in practice are:

Two or more concentrated loads.

Two or more disks or wheels.

Non-uniform distributed loads.

A large number of disks, giving approximately the effect of a distributed load, in which the obliquity must be taken into account.

Disks set at an oblique angle to the center line of the shaft.

Shaft of non-uniform section with various loadings.

Some of these cases have been investigated by Dunkerley, Stodola, Chree, Greenhill, and others. Limitations of space forbid a further consideration of such problems in this place. The reader who wishes to pursue the study of critical speeds further is therefore referred to the works of the authors named.

CHAPTER X

TOOTHED WHEELS

160. Introductory. Rolling Contact.—In many mechanisms motion is transmitted from one link to another by direct rolling contact. Friction gears are examples of such operation and toothed wheels can be shown to be kinematically equivalent to links having pure rolling contact.

Evidently the angular velocities of the links having rolling contact are inversely proportional to the radii from the centers of rotation to the point of contact. It follows that if the velocity ratio is constant the segments into which the line of centers is divided by the point of contact must be constant, and that therefore the rolling curves must have a constant radius.

All contact of this kind may be divided into three classes:

(*a*) Axes parallel.

(*b*) Axes intersecting.

(*c*) Axes neither parallel nor intersecting.

In cases (*b*) and (*c*) the rolling curves are not in the same plane. The angular velocity ratio in these cases is treated in Arts. 162 and 163.

161. Axes Parallel. Rolling Cylinders.—If two cylinders C and D, Figs. 230 and 231, are mounted on two parallel shafts A and B in such a manner that they are kept in contact in all positions, the sum of their radii will equal the distance between the shafts in Fig. 230, while in Fig. 231, the distance between the shafts will equal the difference of the radii. In either case the line of contact is common to both cylinders, and if one cylinder be made to rotate with angular velocity ω, it will drive the other by rolling contact, with angular velocity Ω such that

$$\frac{\omega}{\Omega}=\frac{r}{R}.$$

Note that in Fig. 230 the wheels rotate in opposite directions, while in Fig. 231 they rotate in the same direction.

162. Axes Intersecting. Rolling Cones.—The derivation of rolling cones from rolling cylinders is probably best shown by the use of two right cylinders combined with two right cones as shown in Fig. 232. Each cylinder has one base in common with that of one of the cones, hence the axes of this cylinder and cone must coincide. While the bases of the cones need not be equal the slant heights must be the same. The bases of the two cones have a common tangent in their plane which passes through E and which

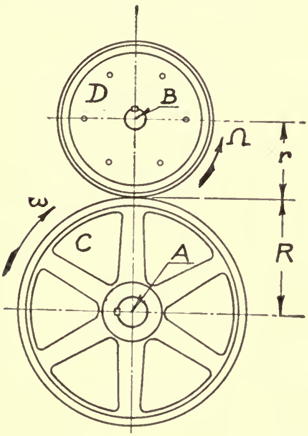


FIG. 230.

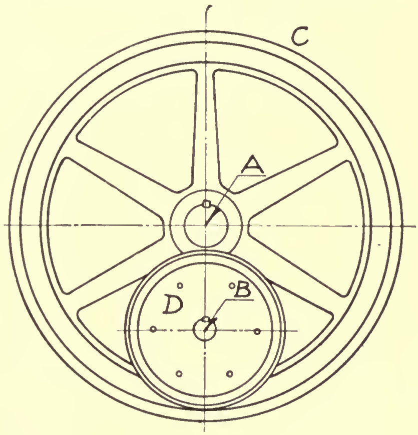


FIG. 231.

is perpendicular to the plane of the paper. If the axis AA be caused to rotate about this tangent, point a can be brought into coincidence with point b and the cones still have a common tangent and will be in contact along element bE, Fig. 233. Hence the cones can roll upon each other in their new position. Evidently the velocities of the two contact points are the same as for the original position.

If from any point F of the line of contact planes are passed perpendicular to the axes AA and BB, these planes will cut circles from the cones which will roll together with the same velocity ratio as the bases.

Hence,

$$\frac{\omega}{\Omega}=\frac{FG}{FH}=\frac{R}{r}, \text{ a constant ratio.}$$

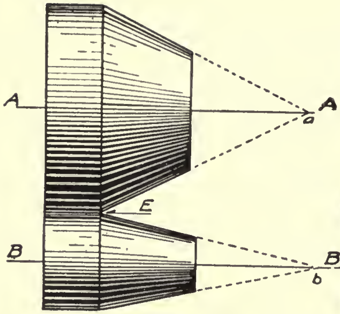


FIG. 232.

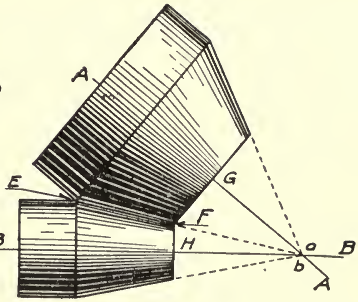


FIG. 233.

Having given the axes, and the velocity ratio for two rolling cones, it is required to construct the cones.

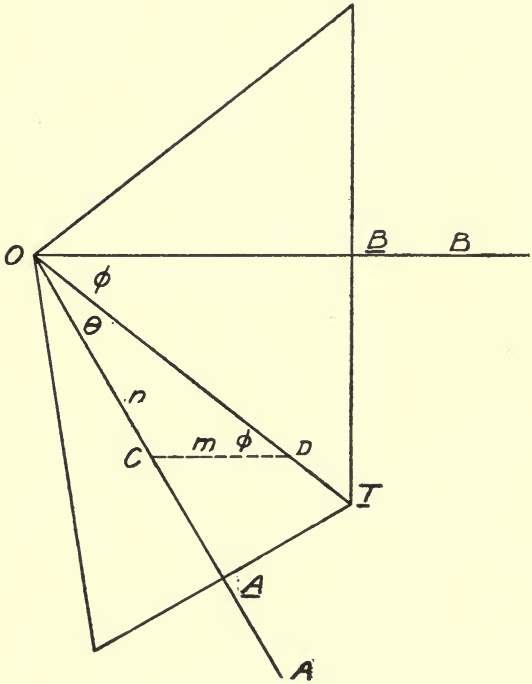


FIG. 234.

In Fig. 234 let OA be the driving axis and OB the following axis; let the velocity ratio of driver to follower be

$$\frac{\omega}{\Omega}=\frac{m}{n}.$$

In other words, OA is to make m revolutions while OB makes n revolutions.

To any convenient scale lay off $OC=n$ divisions. Through C draw CD parallel to OB and let CD be equal in length to m divisions on the same scale. Through D draw ODT, which

will be the line of contact. From any point T of ODT, let fall the perpendiculars AT and BT on the axes. Construct two right cones on OA and OB having AT and BT as radii of their respective bases. These cones will roll together with the required velocity ratio, for:

$$\frac{\omega}{\Omega}=\frac{m}{n}=\frac{\sin\theta}{\sin\phi}=\frac{AT}{OT}\div\frac{BT}{OT}=\frac{AT}{BT}.$$

In other words the radii of the bases have the required constant ratio.

163. Axes Neither Parallel nor Intersecting. Generation of the Hyperboloid of One Sheet.—If any line AA, Fig. 235, be caused to rotate about any other line BB that does not lie in the same plane, the resulting surface is a hyperboloid of revolution. It is evident that a plane through line BB would cut from the surface a hyperbola of which line BB is the conjugate axis.

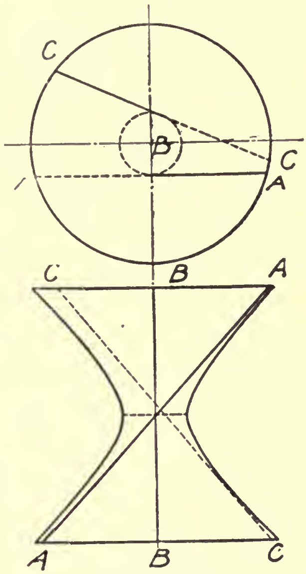


FIG. 235.

The same hyperboloid of revolution would evidently have been generated by the revolution of line CC about line BB as an axis. Hence the hyperboloid of revolution is a double ruled surface.

Sections cut from the surface by planes perpendicular to the axis are circles, the smallest of these being called the *gorge circle*.

Referring to Fig. 236, line XX is a vertical axis and YY is another axis revolved from any position until parallel to a chosen vertical plane but inclined to the horizontal plane. KM is the common perpendicular to the two axes. OC is a straight line parallel to the vertical plane. $o'c'$ makes an angle α with $X'X'$ and an angle β with $Y'Y'$.

If an hyperboloid of revolution be described by the revolution of OC about the vertical axis and another hyperboloid be

described by the revolution of OC about YY as an axis, the two hyperboloids will be in contact along the line OC. If one hyperboloid be made to rotate about its axis with angular velocity

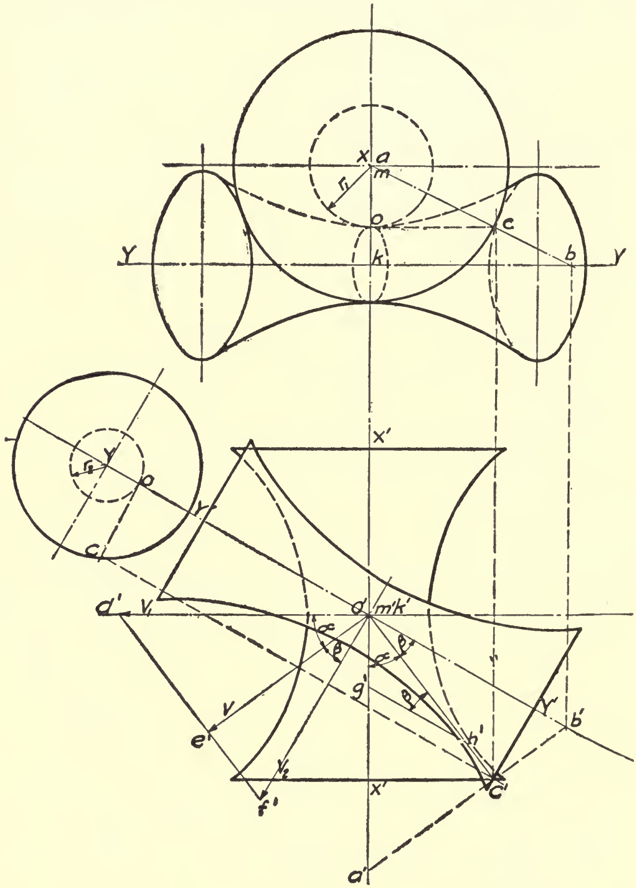


Fig. 236.

ω it will drive the other by rolling contact if the frictional resistance between the hyperboloids is sufficient.

The construction of the hyperboloids is clearly shown in Fig. 236. The end view is a projection of the inclined hyperboloid upon a plane perpendicular to its axis. This view is of service in the layout of the inclined hyperboloid.

To draw a normal to an hyperboloid of revolution from any point on its surface proceed as follows: Through the chosen point draw the two generatrices. The plane containing these two lines will be tangent to the hyperboloid at the chosen point and at this point only. The normal to the surface will be perpendicular to this plane and since the surface is one of revolution, the normal must intersect the axis.

The common normal to the surfaces at C will then be perpendicular to the tangent plane at that point and this tangent plane will contain the line of contact OC. Since the line OC is parallel to the vertical plane the vertical projection $a'b'$ of the common normal must be perpendicular to $o'c'$. But the common normal must intersect the axis of each hyperboloid. Hence $a'c'b'$ at right angles to $o'c'$ is the vertical projection of the common normal. The horizontal projection acb is easily found.

Denote the vertical hyperboloid by the symbol M and the inclined hyperboloid by the symbol N. Let ω and Ω be the angular velocities of M and N respectively. Let V_1 be the linear velocity of the point O considered as a point on the gorge circle of M and let V_2 be the linear velocity of the point O considered as a point on the gorge circle of N. Let r_1 and r_2 be the radii of the gorge circles of M and N respectively.

Draw $o'd'$, $o'f'$ and $o'e'$ at right angles to $X'X'$, $Y'Y'$ and $o'c'$ respectively. Lay off $o'd'$ equal to V_1 and draw $d'e'f'$ perpendicular to $o'e'$, meeting $o'e'$ at e' and $o'f'$ at f'. Since the motion is transmitted by rolling contact the components of V_1 and V_2 perpendicular to OC must be equal. Resolving V_1 and V_2 parallel and perpendicular to OC the components of V_1 are $o'e'$ and $d'e'$, while those of V_2 are $o'e'$ and $e'f'$. In addition to the rolling of the one hyperboloid on the other there is also a sliding motion along the line of contact and the velocity of this sliding is represented by $d'f'$.

Here we have

$$\frac{\omega}{\Omega}=\frac{o'd'}{r_1}\div\frac{o'f'}{r_2}=\frac{o'd'}{o'f'}\frac{r_2}{r_1}.$$

But the triangle $d'o'f'$ is similar to the triangle $o'a'b'$.

Hence

$$\frac{o'd'}{o'f'}=\frac{o'a'}{o'b'}.$$

Also from the top view,

$$\frac{r_2}{r_1}=\frac{cb}{ca} \quad \text{but,} \quad \frac{cb}{ca}=\frac{c'b'}{c'a'}$$

or

$$\frac{\omega}{\Omega}=\frac{o'a'}{o'b'}\times\frac{c'b'}{c'a'}=\frac{c'b'}{o'b'}\div\frac{c'a'}{o'a'}=\frac{\sin\beta}{\sin\alpha}.$$

On $x'o'x'$ make $o'g'$ equal ω and draw $g'h'$ parallel to $Y'Y'$ to meet $o'c'$ at h'. Then angle $o'h'g'=\beta$, and $\frac{o'g'}{g'h'}=\frac{\sin\beta}{\sin\alpha}$, but this is equal to $\frac{r_2}{r_1}$; or, since $o'g'=\omega$, $g'h'$ must be equal to Ω.

Having given the two axes and the relative angular velocities, it is required to find the line of contact and to draw the hyperboloids.

On $x'x'$ lay off $o'g'=\omega$ and draw $g'h'$ parallel to $Y'Y'$ and equal to Ω. Then o' being joined to h', the elevation of the line of contact is found. To find the plan of this line draw $a'c'b'$ perpendicular to $o'c'$ to meet $x'x'$ at a' and $Y'Y'$ at b'. Hence b' is located and by drawing ab we can locate c. Then oc parallel to YY is the horizontal projection and the surfaces may be drawn.

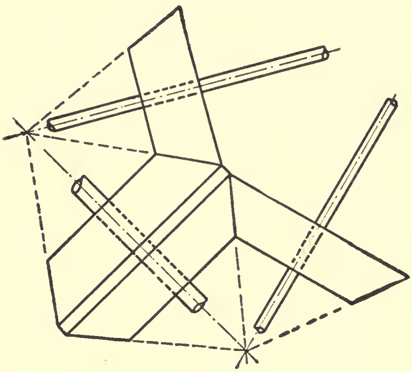


FIG. 237.

If in any hyperboloids the radii of the gorge circles be decreased the meridian curves will become flatter, and the axes in the limit will intersect and the surfaces will be rolling cones.

If the axes are neither parallel nor intersecting, it is possible to transmit motion from one to the other by pure rolling contact if two pairs of rolling cones are used as shown in Fig. 237.

164. Friction Gearing.—Friction gearing is employed where a constant velocity ratio is not imperative. When working without slip friction gears transmit precisely the same motion as toothed wheels. The instantaneous center of relative motion is

at the point of contact if there is no slippage, but becomes indeterminate in case slipping exists. In the latter case the motion is unconstrained, and the drive is not positive. Hence in dealing with toothless wheels from the standpoint of kinematics, it is usually assumed that there is no slippage. The material and construction of the wheels together with the pressure with which they are held in contact will determine the magnitude of the force that can be transmitted.

In many applications the possibility of slip is desirable. For example, in hoisting machinery if the car or skip strikes an obstruction while being hoisted or if the drum is overwound the slippage of the friction drive lessens the danger of breakage of parts of the machine.

From the standpoint of kinematics, any of the surfaces discussed in Arts. 161 to 163 will roll together and may be used as friction gears. Practically, rolling cylinders and rolling cones or special cases of the latter are most common.

165. Spur Frictions. Grooved Spur Frictions.—The simplest form of friction gearing consists of two plain cylinders, rotating on parallel shafts and held together by properly constructed bearings, Figs. 230 or 231.

Such gearing is used for light power hoists, coal screens, friction-board drop hammers, etc.

If the amount of power to be transmitted is increased, the pressure holding the rolls together must be increased. This pressure may be excessive upon the shaft and cause a considerable loss of power due to journal friction. To decrease this loss, a form of gearing known as grooved spur friction wheels, Fig. 238, is used.

166. Beveled Friction Gearing. Crown Friction Gearing.—Beveled friction gearing, Fig. 239, is used where it is desired to transmit power between intersecting shafts.

If the shafts intersect at 90°, crown friction gearing is often used to transmit power, as shown in Fig. 240. As usually installed disk *b* acts as the driver and wheel *a* as the follower, although from the point of view of uniformity of wear the reverse arrangement is preferable. The wheel *a* is so mounted that it can be drawn across the face of the disk *b*, thus varying the velocity ratio and also the rotational direction of the driven shaft.

167. Spur Gearing.—Where the velocity ratio between the driving and driven members must be absolutely positive, or when the power to be transmitted is large, it becomes necessary to provide the contact surfaces with grooves and projections, or teeth,

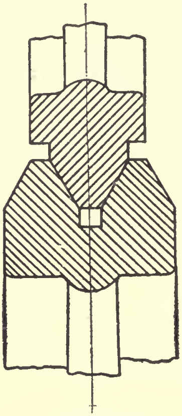


FIG. 238.

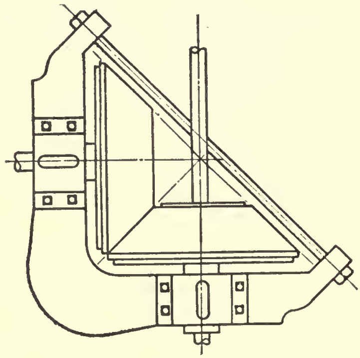


FIG. 239.

thus providing a positive means of rotation. The rolling surfaces are then called *Pitch Surfaces*, and sections of them perpendicular to their axes are called *Pitch Circles*. The point of tangency of these circles is called the *pitch point*.

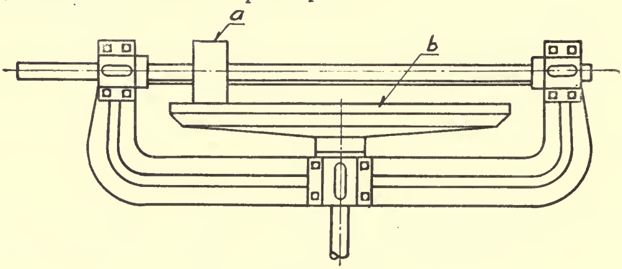


FIG. 240.

The teeth must have such a form as to satisfy the following conditions:

1. For satisfactory operation and for the transmission of a uniform velocity ratio the teeth must have a form such that the common normal will always pass through the pitch point.

2. Friction and wear attend sliding contact. Hence the relative motions of the teeth should be as much a rolling motion as possible. It is impossible to have a pure rolling contact with toothed gearing and at the same time a constant velocity ratio.
3. The teeth should be symmetrical on both sides in order that the gear may run in either direction.
4. The *arc of action* should be long enough to insure the meshing of more than one pair of teeth.

A number of shapes of teeth will satisfy these conditions, but, the two commonly used are the involute and cycloidal teeth, so called, because of the curves on which they are based.

168. Definitions.—As a preliminary to the discussion of the forms of gear teeth it is necessary to define certain terms which are used in connection with gears.

(*a*) The *circular pitch* is the distance measured along the pitch circle from a point on one tooth to the corresponding point on the next tooth, or the circumference of the pitch circle divided by the number of teeth.

(*b*) The *chordal pitch* is the distance measured on the *chord* of the pitch circle from a point on one tooth to the corresponding point on the next tooth.[1]

(*c*) The *diametral pitch* is the ratio of the number of teeth in the gear to the pitch diameter expressed in inches. It should be noted that this is not a dimension, but simply a convenient ratio.

(*d*) The *thickness of the tooth* is the thickness measured on the arc of the pitch circle. See Fig. 241.

(*e*) The *tooth space* is the width, measured on the arc of the pitch circle, of the space between two adjoining teeth.

(*f*) The *backlash* is the difference between the tooth space and the thickness of the tooth.

(*g*) The addendum is the distance from the pitch circle

[1] The chordal pitch is used only in the layout of a drawing or by the pattern maker in forming the teeth on a wooden pattern.

to the outer ends of the teeth. It is shown as dimension *a* in Fig. 241.

(*h*) The *addendum circle* is the circle bounding the outer ends of the teeth.

(*i*) The *dedendum* is the distance from the pitch circle to the bottom of the tooth space. It is shown as dimension *b* in Fig. 241.

(*j*) The *dedendum circle* is the circle bounding the bottom of the tooth spaces.

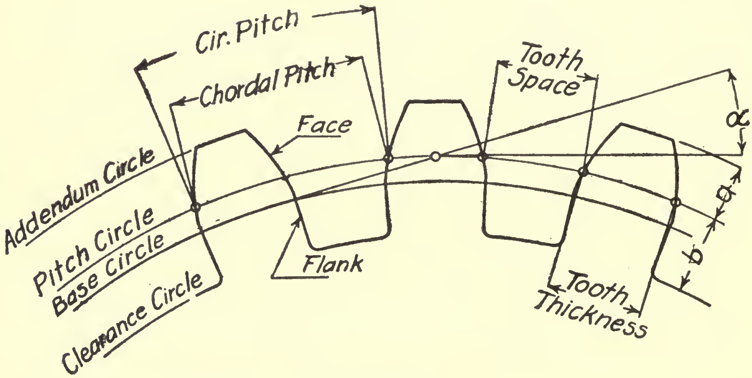


FIG. 241.

(*k*) *Clearance* is the amount of space measured on the line of centers between the addendum circle of one gear and the dedendum circle of the mating gear.

(*l*) The *face of the tooth* is that part of the tooth profile lying between the pitch circle and the addendum circle.

(*m*) The *flank of the tooth* is that part of the tooth profile lying between the pitch circle and the dedendum circle.

(*n*) The *base circle* is an imaginary circle used in *involute gearing* to generate the involutes which form the tooth profiles. It is drawn tangent to the line of action of the tooth thrust.

(*o*) The *describing circle* is an imaginary circle used in laying out cycloidal gearing. There are two such

circles, usually of the same size, one to generate the epicycloid, which forms the face of the tooth, and the other to generate the hypocycloid which forms the flank of the tooth.

(*p*) The *angle of obliquity* is the angle formed by the line of action of the pressure between a pair of mating teeth and a tangent to the pitch circle drawn through the pitch point. This angle is represented by α in Fig. 241.

(*q*) The *arc of approach* is the arc measured on the pitch circle from the pitch point to the position of the tooth at which contact begins, as shown by AC or BC, Fig. 242.

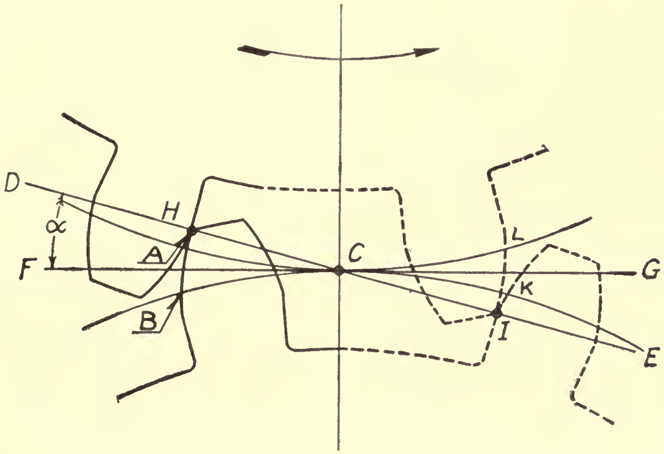


Fig. 242.

(*r*) The *arc of recess* is the arc measured on the pitch circle from the pitch point to the position of the tooth at which contact ends, as shown by CK or CL, Fig. 242.

(*s*) The *arc of action* is the sum of the arcs of approach and recess.

(*t*) The *velocity ratio* is the ratio of the number of revolutions of the driver to the number of revolutions of the driven gear.

169. Relation between Circular and Diametral Pitch.—By definition:

$$\text{Circular Pitch} = \frac{\text{Circumference of Pitch Circle}}{\text{Number of teeth}} = \frac{2\pi r}{t} = \frac{\pi d}{t} = p'.$$

$$\text{Diametral Pitch} = \frac{t}{d} = p.$$

Hence,

$$pp' = \frac{\pi d}{t} \times \frac{t}{d} = \pi.$$

That is, the product of the circular pitch times the diametral pitch is equal to π.

170. Rectification of Circular Arcs.—In laying out gears it is frequently necessary to lay off a line equal in length to a given arc. The best geometrical constructions for this purpose are those due to Rankin (A Manual for Machinery and Mill Work), and are as follows:

(*a*) To draw a straight line approximately equal to a given circular arc AB, Fig. 243.

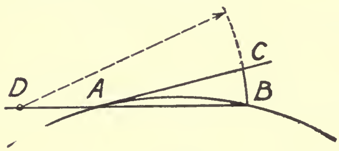


FIG. 243.

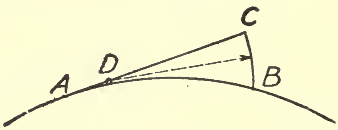


FIG. 244.

Join BA and produce it to D making $AD = \frac{1}{2}\ AB$. With center D and radius DB describe the arc BC cutting the tangent AC at C. AC is the straight line required. If the given arc subtends an angle greater than 60° it should be subdivided.

The error varies as the fourth power of the angle AOB, where O is the center of the circle of which AB is an arc. When the angle AOB is 30°, AC is less than the arc AB by about $\frac{1}{14400}$ of the length of the arc.

(*b*) To lay off on a given circle an arc AB approximately equal to a given length, AC, Fig. 244.

Draw a tangent AC to the circle at A, and make AC equal to the given length. Make $AD = \frac{1}{4}\ AC$. With center D and

radius DC describe the arc CB to cut the circle at B. AB is the arc required.

The error in (*b*) expressed as a fraction of the given length is the same as in (*a*) and follows the same law.

171. Generation of Cycloidal Curves.—A *cycloid* is the curve described by a point on the circumference of a circle which rolls on a straight line, the circle and straight line remaining in the same plane.

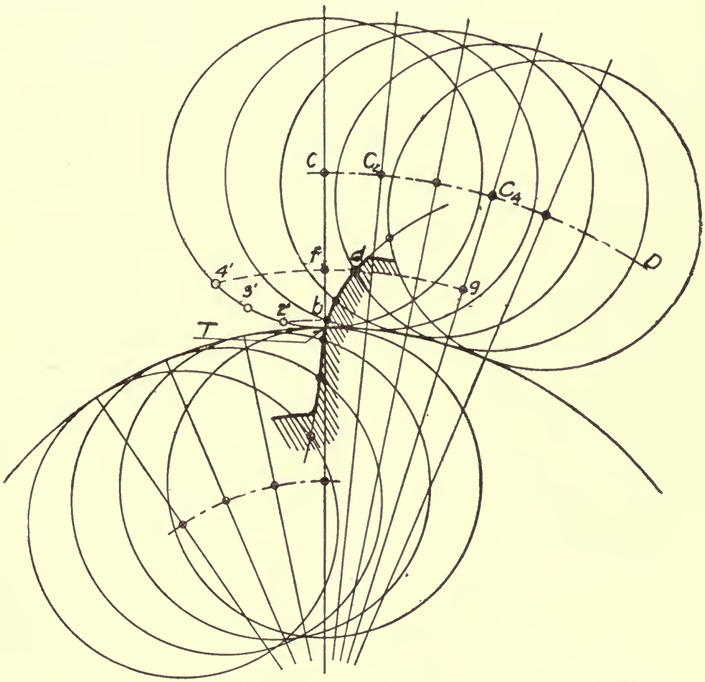


FIG. 245.

If the circle be rolled on the outside of another circle, the two circles remaining in the same plane, the curve described is called an *Epicycloid;* when rolled inside the curve described is called a *Hypocycloid.*

(*a*) To draw an epicycloid, Fig. 245, let

$$r_1 = \text{the radius of the rolling circle;}$$

$$r_2 = \text{the radius of the fixed circle.}$$

Draw through C a circle CD concentric with the fixed circle. Evidently, the radius of this circle will equal r_1+r_2 and the center of the describing circle will always be on CD. Divide the describing circle into a convenient number of parts and lay off these lengths (Art. 170) on the circumference of the fixed circle. Through the latter points of division draw radii and produce them to intersect the line of centers CD. The points thus located will be the successive positions of the center C of the describing circle.

If the describing circle is drawn in any one of these positions, as C_2, its intersection b with the circular arc through $2'$ will be a point on the required curve. Other points will be determined in the same way.

A simpler and more accurate method for constructing the curve is to produce the arc $f4'$ to intersect the radius through $C4$ at g and to lay off $gd=f4'$. Other points will be determined in the same way.

This method gives the best results for points of the curve near T which is the part of the curve employed in the construction of gear teeth.

(*b*) The construction of the *Hypocycloid* is the same as that for the epicycloid, except that the radius for the line of centers is here r_2-r_1.

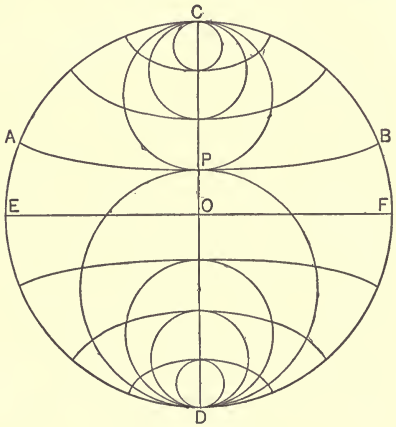


FIG. 246.

A given hypocycloid on a given base circle may be described by a point on the circumference of either of two rolling circles. In Fig. 246 APB is the hypocycloid described by point P of the rolling circle CP when rolled on the fixed circle $ACBFDE$. The same curve would be described by the point P considered as a point of the rolling circle DP.

An important case of the hypocycloid is that where the diameter of the rolling circle is equal to the radius of the fixed circle.

In this case the hypocycloid becomes a straight line *EOF*, which is a diameter of the circle.

(*c*) The construction of the *Cycloid* is the same as that for the epicycloid except that the center of the fixed circle is moved to infinity and the circumference becomes a straight line, Fig. 247.

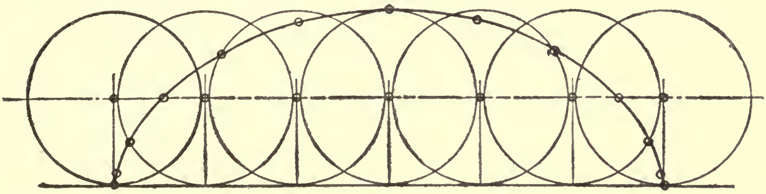


FIG. 247.

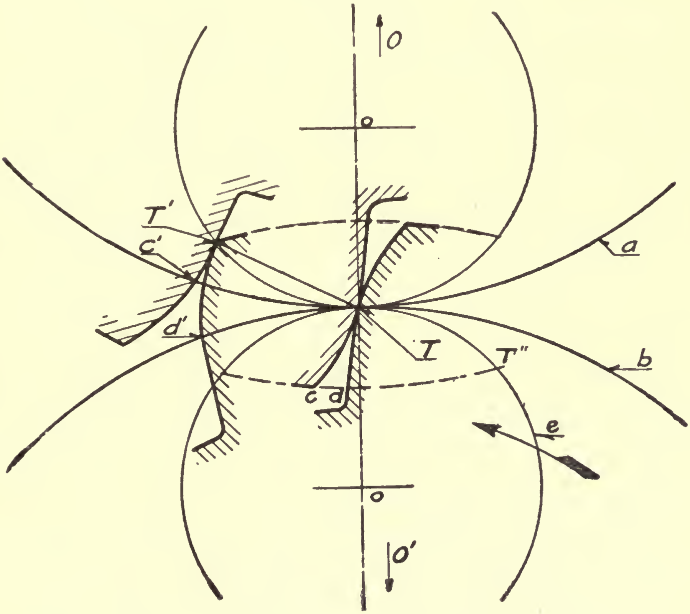


FIG. 248.

In Fig. 248 *a* and *b* are two pitch circles with centers at *O* and *O′* respectively. The velocity ratio is then $\frac{\omega}{\Omega}=\frac{O'T}{OT}$. Let *oT*

be the radius of any describing circle e such that oT is less than $O'T$. By rolling this circle on the outside of circle a describe the epicycloid Tc and by rolling it inside of circle b describe hypocycloid Td. Let the circles a, b, and c rotate about their respective centers in rolling contact.

At any instant the curves $T'c'$ and $T'd'$ will be in contact at the point T' in the circumference of the describing circle since by construction,

$$\text{arc } TT' = \text{arc } Tc' = \text{arc } Td'.$$

Referring to Fig. 249 it is evident that when a circle rolls on the outside of another circle in the same plane the instantaneous

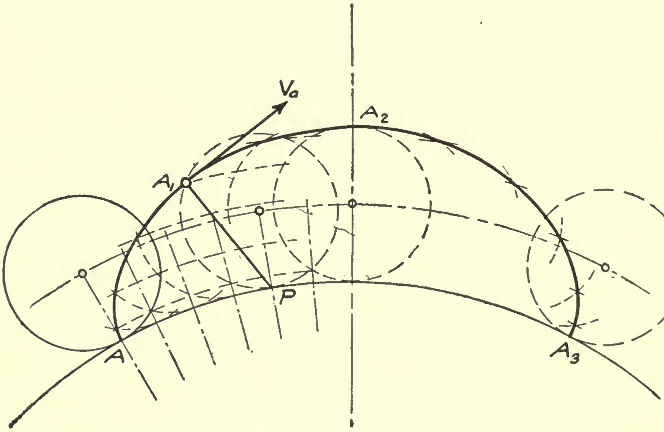


Fig. 249.

center is the point of contact and hence the normal passes through this point. The same is true for the case of the hypocycloid. Therefore, the curves must have a common normal and a common tangent at the point T', Fig. 248.

In Fig. 248 the instantaneous center is the point T and the common normal is the line $T'T$ at that instant. The same proof can be repeated for any other position of the two curves. It will be noted that the common normal always cuts the line of centers at the point T, which is the pitch point. Hence these curves will transmit a motion which will exactly replace the rolling action of the pitch circles.

If the diameter of the describing circle were made equal to

the radius of the fixed circle, the hypocycloid would become a straight line, but the two curves would work together as before. The proof of this statement is left to the student.

If the diameter of the describing circle is still further increased, the curve generated becomes convex in a direction opposite to that of the epicycloid, and it lies on the opposite side of the line of centers. In the limit where the diameter of the describing circle is equal to the diameter of the pitch circle the hypocycloid will have become a point. This point is the basis for *pin gearing* and may be driven by the epicycloid with the same velocity ratio as the pitch circles.

Referring to Fig. 248, if the circles be rolled together, the point of contact T' must follow the arc $T'T$ until points T', c' and d' coincide at T and, if the motion is continued, it must follow the arc TT'', to the point where contact ceases.

172. Teeth of Wheels.—In Fig. 250, O_1 and O_2 are the centers, and a and b are the pitch circles of the driver and driven wheel; o_1 and o_2 are the centers of the describing circles whose diameters are less than the respective radii of the pitch circles in which they roll. Let $Tc' = Td' =$ the pitch and let T' be the point where the teeth quit contact and which is therefore a point on the addendum circle.

If the circles are caused to turn about their respective centers in rolling contact the point T will trace the epicycloid Te, which will be the face of the driver's tooth, and the hypocycloid Tf, which will be the flank of the follower's tooth.

To complete the addendum of the driver's tooth draw the addendum circle through T'; bisect Td' at m and through m draw an epicycloid reversed in direction to intersect the addendum circle at n.

Pn is the part of the flank of the follower that comes in contact with the epicycloid mn, but in order to make room for the point of the driver's tooth, as it revolves, Pn is continued by the amount of the clearance, until the root circle is met. The profile is usually joined to the root circle by a small circular arc or fillet.

The face of the follower's teeth and the flank of the driver's teeth are determined by laying off T'', the point where contact begins, and proceeding as before.

The complete wheels can then be laid out by dividing both pitch circles into arcs equal to one-half the pitch, and by then drawing curves similar to those already found in their proper order through these points.

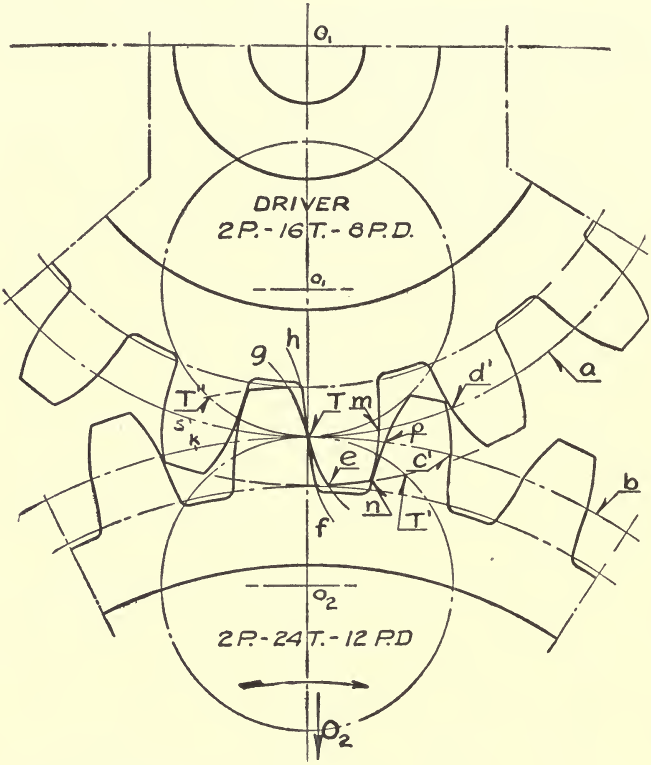


Fig. 250.

(*a*) The arc of approach determines the addendum of the follower's tooth or vice versa.

(*b*) The arc of recess determines the addendum of the driver's tooth or vice versa.

173. Relation between Pitch and Arcs of Approach and Recess.—Having given the diameters of the pitch and describing circles and the arcs of approach and recess, it is required to determine the limits between which the pitch may vary.

Let T' and T'', Fig. 251, be the points where contact between the mating teeth begins and ends. Professor Willis has pointed out that if the radial line O_1T' be drawn, the distance gc can nerver be greater than one-half the thickness of the tooth; in other words, the pitch must not be less than four times gc. If the pitch is made equal to four times gc the tooth will become pointed as indicated at rs. It is also evident that the pitch must not be greater than the arc of action, for in that case contin-

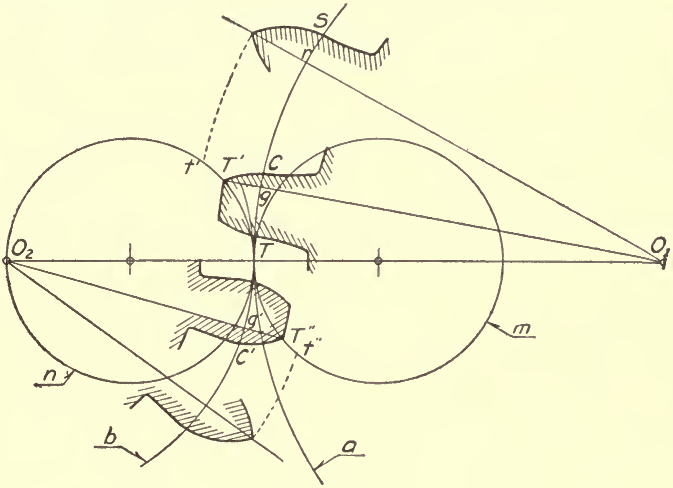


FIG. 251.

uous contact becomes impossible. The upper and lower limits of the pitch are therefore given by the inequality,

$$4gc \leq p' \leq \text{arc of action.}$$

174. Example.—

Let the radius of gear a, Fig. 250, be $2\frac{1}{2}$ inches;
the radius of gear b, Fig. 250, be $1\frac{1}{2}$ inches;
the radius of the describing circles be $\frac{3}{4}$ inch;
the arc of approach $= TT' = \frac{1}{2}$ inch;
the arc of recess $= TT'' = \frac{5}{8}$ inch.

Constructing the teeth with these data, cg is found by measurement to be $\frac{11}{64}$ inch, and $c'g'$ is found to be $\frac{13}{64}$ inch. Therefore

the pitch cannot be greater than $\frac{5}{8}+\frac{1}{2}=1\frac{1}{8}$ inches or less than $4\times\frac{13}{64}=\frac{13}{16}$ inch. Both of these limits should be avoided. The pitch of, course, must divide both pitch circles evenly.

175. Size of Describing Circle.—It is evident that a describing circle of any size, smaller than the pitch circle in which it rolls, may be used to produce a curve fulfilling the required conditions.

In Fig. 246 it was shown that, when the diameter of the describing circle is equal to the radius of the pitch circle, the hypocycloid described becomes a radial line in the pitch circle. This condition is undesirable from the standpoint of the strength of the teeth.

It follows that it is desirable to have the diameter of the describing circle less than the radius of the pitch circle. This condition gives the teeth spreading flanks which are much stronger. Where but two wheels are to mesh together, the diameter of the describing circle is usually made three-eighths the diameter of the pitch circle in which it rolls.

It is evident that, for a given pitch, the length of teeth required for a given arc of action is less, as the describing circles are made greater in diameter.

During the contact the face of the teeth on one wheel acts only on the flank of the teeth on the other. Hence the form of the face on one wheel does not effect that of its own flanks. Therefore there is no fixed relation as to size between the describing circles used in the two wheels.

176. Interchangeable Wheels.—In making a set of patterns or making cutters for cut teeth, it is desirable that any gear of a set shall work with any other gear of the same pitch. When this is the case the set is interchangeable.

The conditions necessary in an interchangeable set of gears are that all of the wheels of the set shall have the same pitch and that all of the tooth curves shall be generated by the same describing circle. The size of the describing circle usually used is such that the smallest gear of the set shall have radial flanks. This gear is generally taken as having from 12 to 15 teeth. The diameter of the describing circle would then be equal to $\frac{tp'}{2\pi}$, where t = the number of teeth in the smallest gear.

177. Example.—Fig. 252, drawn half size, is the practical solution of the following problem, assuming cast teeth:

Distance between shaft centers = 12 inches.
Driving shaft turns at 200 r.p.m.
Driven shaft turns at 400 r.p.m.

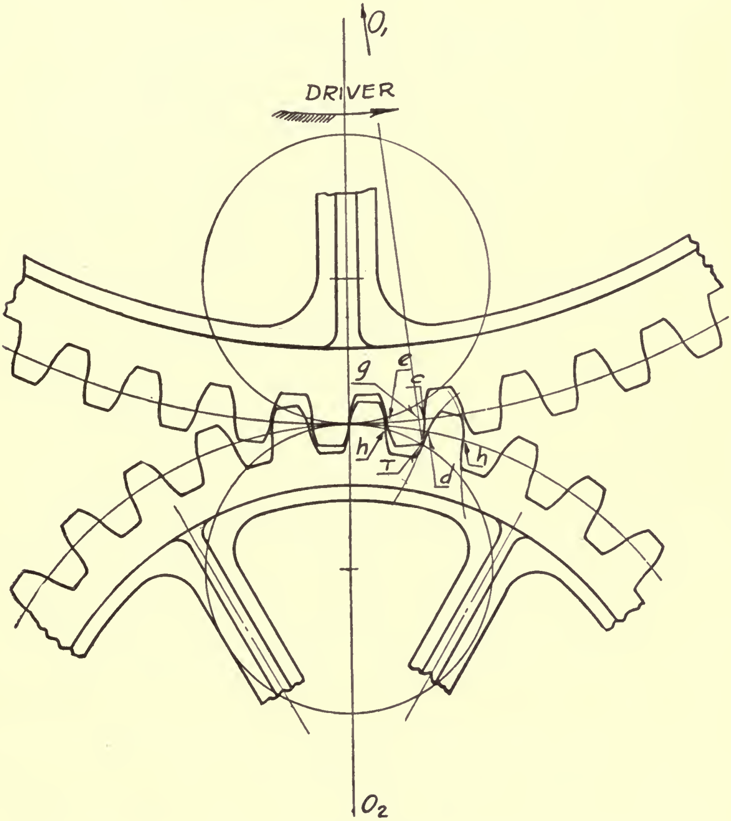


FIG. 252.

Diametral pitch = 4.
Diameter of describing circle = $\frac{3}{8}$ × diameter of smaller pitch circle.

$$r_1 = \frac{400}{200+400} \times 12 = 8 \text{ inches rad.}$$

$$r_2 = \frac{200}{200+400} \times 12 = 4 \text{ inches rad.}$$

Size of describing circle $= \frac{3}{8} \times 8 = 3$ inches diameter.
Number of teeth in gear $= 4 \times 16 = 64$.
Number of teeth in pinion $= 4 \times 8 = 32$.

Circular pitch $= \dfrac{\pi}{4} = 0.7854$ inch.

From Art. 197:

Addendum $= 0.3p' = 0.3 \times 0.7854 = 0.2356$ inch.
Dedendum $= 0.4p' = 0.4 \times 0.7854 = 0.31416$ inch.
Radius of addendum circle (gear) $= 8 + 0.2356 = 8.2356$.
Radius of addendum circle (pinion) $= 4 + 0.2356 = 4.2356$.
Radius of dedendum circle (gear) $= 8 - 0.314 = 7.686$.
Radius of dedendum circle (pinion) $= 4 - 0.314 = 3.686$.

Lay out the pitch circles and describing circles making the lower wheel the pinion.

Draw the addendum circle of the gear, thus locating T' where the teeth will quit contact. Draw the dedendum circle of the pinion. Through T' lay down the epicycloid $T'c$. On the pitch circle of the gear, lay off ce = one-half the pitch. Draw the radial line O_1T', intersecting the pitch circle at g, then since gc is less than one-half ce, the case is a practical one. Draw through e an epicycloid reversed in direction, thus forming the addendum for one tooth. Lay down the hypocycloid $T'd$, and continue it to the dedendum circle, joining the two by a small fillet. On the pitch circle of the pinion, lay off dh = one-half the pitch, and draw through h a hypocycloid reversed in direction, joining it to the dedendum circle as before, thus forming the dedendum for one tooth.

In the same way, the teeth of the driven wheel may be proven satisfactory. The addendum of its teeth, and the dedendum of the driver's teeth are next determined.

The pitch points of the teeth are then laid off on the pitch circles, and proper curves, bounded by the addendum and dedendum circles, are drawn in.

178. Rack and Pinion.—If we consider the center of one of the gears just described as being moved to infinity, the case of the rack and pinion, Fig. 253, would result. Thus it is evident that all of the deductions of the previous articles apply equally well, with obvious modifications, to this case. The faces and

flanks of the rack are cycloids; and in an interchangeable set of wheels they are alike. In this case, any gear of the set will work with the rack. The motion of the rack must be reciprocating.

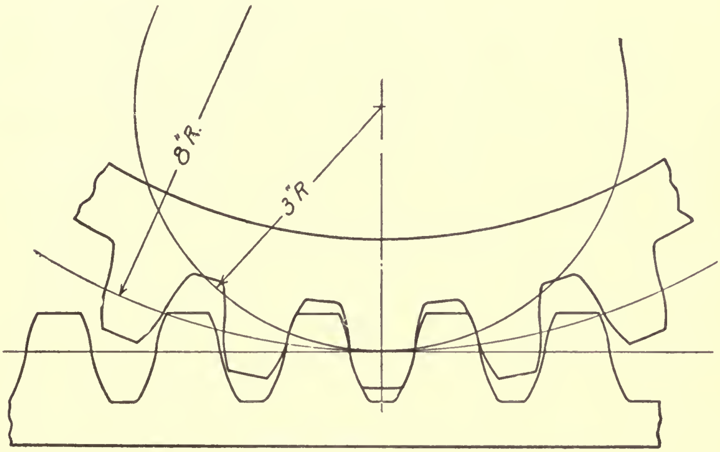


Fig. 253.

179. Annular Gears.—The theory of the forms and the method of drawing the tooth outlines for annular wheels, Fig. 254, are

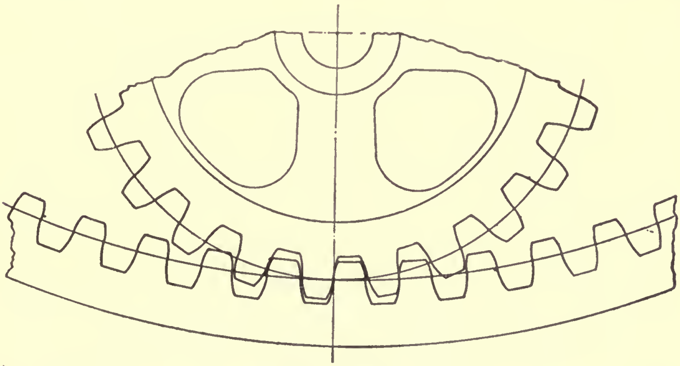


Fig. 254.

the same as for spur gears. Here, however, the faces of the pinion teeth, and the flanks of the gear teeth are both epicycloids, while

the flanks of the pinion teeth and the faces of the gear teeth are both hypocycloids.

180. Approximate Methods for Laying-out the Cycloidal Tooth.—A number of simple methods for the approximate lay-out of tooth forms, represented by circular arcs, have been devised. Those due to Professor Willis and to Mr. George B. Grant are most generally used. These circular arcs, as will be seen, do not follow the exact tooth form, but gears with cast teeth, especially if the pitch is small, also depart from the ideal form.

In the case of cut gears, the exact outline is usually used in making the cutters. The cutters are also made automatically.

The method of layout devised by Professor Willis gives circular arc tooth profiles which are correct for one point only, and which have radii equal to the mean radius of curvature of the exact curve. In this system the approximate faces lie entirely within the true epicycloid.

In the method of layout devised by Mr. Grant, called by him the "Three-point Odontograph," the circular arcs pass through three important points of the exact outlines of the faces. These points are, at the pitch line; the addendum line and a point midway between. Thus the arc devised by Mr. Grant crosses the true curve twice, and, as the average error is much less than that of the arc devised by Professor Willis, it is to be preferred. The student should draw an exact tooth profile to a large scale, and superimpose an approximate profile on it for comparison.

181. Grant's Odontograph for Cycloidal Gears.—In Tables I and II, which follow, are given the radii for cycloidal teeth as worked out by Mr. G. B. Grant. The radii of the circular arcs and the radial distances, from their lines of centers to the pitch line, as given, are for *One Diametral Pitch* or for *One-Inch Circular Pitch.* The table corresponding to the kind of pitch adopted should be used.

In laying out the profiles of cycloidal teeth, by circular arcs, Fig. 255, draw the pitch, addendum, and clearance circles and lay off the pitch of the teeth on the pitch circle, dividing the latter properly for the tooth thickness and tooth space. Draw the line of face centers, circle *B*, at a distance *a*, inside of the pitch line, and the line of flank centers, circle *C*, at a distance *e* outside of the pitch line.

TABLE I

Number of Teeth		Divide by the Diametral Pitch			
Exact	Approx.	Rad. *b*	Dist. *a*	Rad. *c*	Dist. *e*
10	10	1.99	0.02	− 8.00	4.00
11	11	2.00	0.04	−11.05	6.50
12	12	2.01	0.06	∞	∞
13½	13–14	2.04	0.07	15.10	9.43
15½	15–16	2.10	0.09	7.86	3.46
17½	17–18	2.14	0.11	6.13	2.20
20	19–21	2.20	0.13	5.12	1.57
23	22–24	2.26	0.15	4.50	1.13
27	25–29	2.33	0.16	4.10	0.96
33	30–36	2.40	0.19	3.80	0.72
42	37–48	2.48	0.22	3.52	0.63
58	49–72	2.60	0.25	3.33	0.54
97	73–144	2.83	0.28	3.14	0.44
290	145–300	2.92	0.31	3.00	0.38
Rack	......	2.96	0.34	2.96	0.34

TABLE II

Number of Teeth		Multiply by the Circular Pitch			
Exact	Approx.	Rad. *b*	Dist. *a*	Rad. *c*	Dist. *e*
10	10	0.62	0.01	−2.55	1.27
11	11	0.63	0.01	−3.34	2.07
12	12	0.64	0.02	∞	∞
13½	13–14	0.65	0.02	4.80	3.00
15½	15–16	0.67	0.03	2.50	1.10
17½	17–18	0.68	0.04	1.95	0.70
20	19–21	0.70	0.04	1.63	0.50
23	22–24	0.72	0.05	1.43	0.36
27	25–29	0.74	0.05	1.30	0.29
33	30–36	0.76	0.06	1.20	0.23
42	37–48	0.79	0.07	1.12	0.20
58	49–72	0.83	0.08	1.06	0.17
97	73–144	0.90	0.09	1.00	0.14
290	145–300	0.93	0.10	0.95	0.12
Rack	......	0.94	0.11	0.94	0.11

Fig. 255 drawn half size is the solution of the following problem; assuming Brown & Sharpe cut teeth:

Number of teeth = 40.
Diametral pitch = 2.
Diameter of pitch circle = $\frac{40}{2}$ = 20 inches.

From Table VI, Art. 198:

Length of addendum = $0.3183p' = 0.3183 \times 1.5708 = 0.501$ in.
Length of dedendum = $0.3683p' = 0.3683 \times 1.5708 = 0.579$ in.
Radius of addendum circle = 10 in. + 0.501 = 10.501 in.
Radius of dedendum circle = 10 in. − 0.579 = 9.421 in.

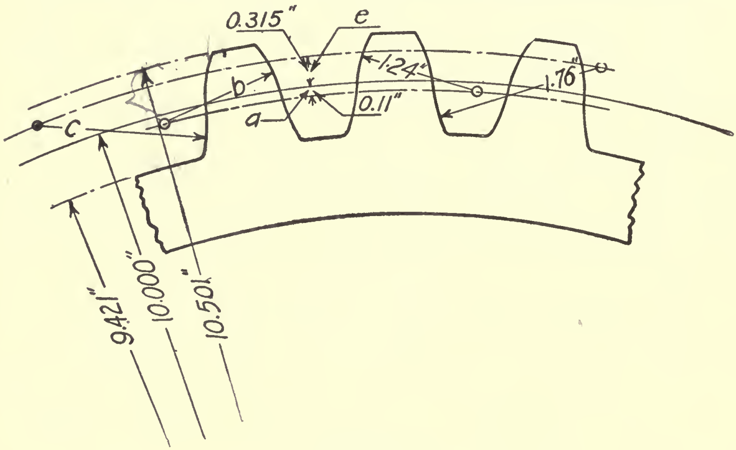


Fig. 255.

From Table I, opposite 40 teeth:

$$\text{Distance } a = \frac{0.22}{\text{Diametral pitch}} = \frac{0.22}{2} = 0.11 \text{ inch.}$$

$$\text{Distance } c = \frac{0.63}{\text{Diametral pitch}} = \frac{0.63}{2} = 0.315 \text{ inch.}$$

All tooth faces are circular arcs with centers on circle *B*.
All tooth flanks are circular arcs with centers on circle *C*.

$$\text{Radius } b = \frac{2.48}{\text{Diametral pitch}} = \frac{2.48}{2} = 1.24 \text{ inches.}$$

$$\text{Radius } c = \frac{3.52}{\text{Diametral pitch}} = \frac{3.52}{2} = 1.76 \text{ inches.}$$

With these radii and centers on their respective lines of centers, draw circle arcs through the pitch points. Terminate the tooth profiles at the addendum and dedendum circles, and put in fillets at the bottoms of the spaces.

If the circular pitch is used, the construction is similar to that just described, except that the values in Table II must be multiplied by the circular pitch *in inches.*

The smallest gear in the set as here given is one having ten teeth, while the smallest one for which standard cutters are manufactured is one having 12 teeth.

182. Involute System. The Involute Curve.—The curve most commonly used for tooth profiles is the involute of a circle. This curve is the path traced by a point on a line as the line rolls on a fixed circle. It may also be described as the path traced by a point at the end of a string, as the string is unwound from a circle, the string being kept taut at all times. Thus in Fig. 256 it will be seen that the string or line will be tangent to the circle in all positions, and that the length to be laid off on the tangents must be equal to the length of the arc (rectified) between the point of tangency and the starting point P.

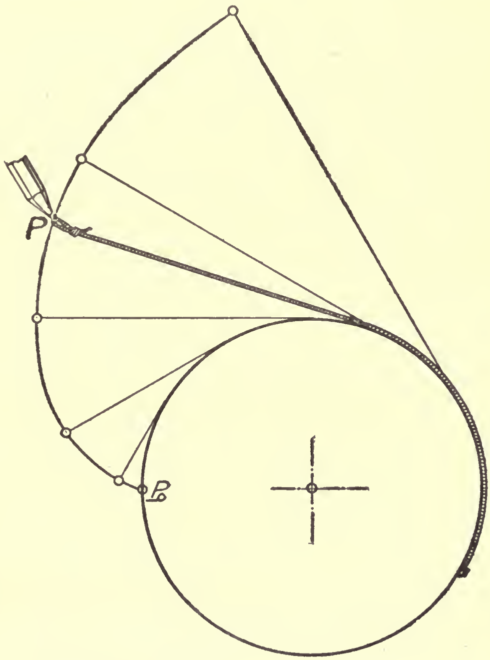


FIG. 256.

It is evident that the involute is a special case of the epicycloid where the center of the describing circle is moved to infinity, and

that, therefore, the point of tangency is the instantaneous center of motion for the point P in any position.

It is to be noted that:

(*a*) Any tangent to a circle is a normal to any involute of that circle.

(*b*) The direction of the velocity of any point P on the involute is perpendicular to the normal drawn from P tangent to the circle.

183. Involutes in Sliding Contact.—In Fig. 257 a and b are two pitch circles with centers at O_1 and O_2 respectively. The velocity ratio is then $\frac{\omega}{\Omega}=\frac{O_2T}{O_1T}$. Draw the common tangent tt', and through T draw a straight line making any angle α with tt'. From the centers O_1 and O_2 drop the perpendicular O_1E and O_2F upon the inclined line ETF. With O_1E and O_2F as radii describe the circles a' and b' which by construction must be tangent to the line ETF, and which will be the base circles for the involutes to be used. Two involutes of the circle a' and b' are tangent at T.

An instant later the same involutes will be in contact at T'. In Art. 182 it was shown that the normal to one of the involutes at the point of contact will be a tangent to the circle a' drawn through the point of contact, and that the normal to the other involute will be a line drawn from the same point tangent to the circle b'.

Since the curves are tangent at the point T' they have a common tangent, and hence the normals must coincide in direction. This normal, however, must be tangent to both base circles a' and b', and must therefore be the line ETF. As this common normal cuts the line of centers in the pitch point T, the velocity ratio is constant.

184. Path of Point of Contact.—Referring to Fig. 257, if the pitch circles be rolled together, the point of contact T' must follow the straight line ETF until points T', d, and f coincide at T, and if the motion is continued, it must follow the same line to some point T'' where contact ceases.

185. Teeth of Wheels.—In Fig. 257, O_1 and O_2 are the centers, and a and b are the pitch circles of the driver and driven wheel.

tTt' is their common tangent and ETF is a straight line making an angle $t'TE=\alpha$ with the common tangent. By definition α is the angle of obliquity. It should be noted that in standard practice the angle of obliquity is made $14\frac{1}{2}°$. This angle was probably chosen because the sine of the angle equals 0.25, a proportion that was easy for the millwright to lay out at the time when all gears were cast. a' and b' are the base circles for the

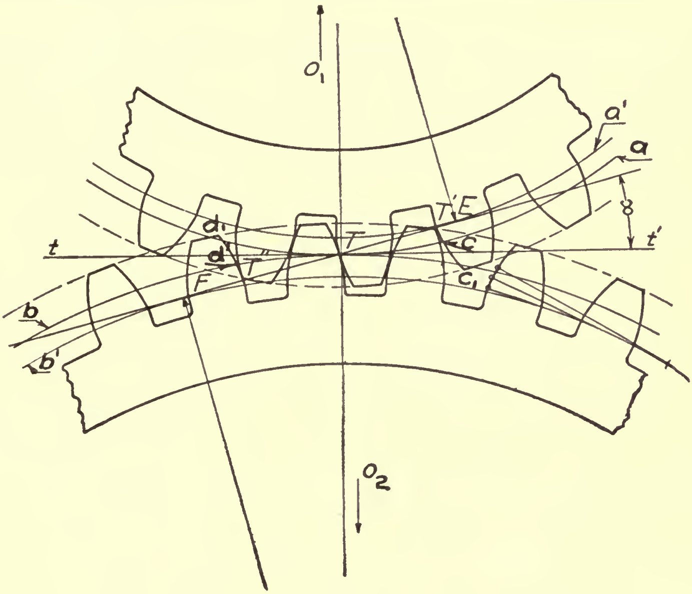


Fig. 257.

profiles, drawn with radii O_1E and O_2F, O_1E and O_2F being perpendicular to the line of obliquity of action ETF. Let $Tc=Td'$ =the pitch. On the base circle a' draw the involute c_1cT', intersecting the line ETF at T', and on the base circle b' draw the involute $d_1d'T''$, intersecting the line ETF at T''. Here $T'c_1$ is the profile of the driver's tooth and $T''d_1$ is the profile of the driven tooth.

It is evident that:

(*a*) Clearance must be allowed for the points of the teeth as they revolve.

(*b*) The involutes cannot extend within the base circle from which they are derived. Radial lines are used to complete the profiles inside of these circles, these lines being joined to the dedendum circles by small circular arcs.

Draw the addendum circles through points T' and T'' and draw the dedendum circles after making allowance for the clearance.

Contact begins at T'' and ends at T'. The complete wheels can then be laid out by laying off half the pitch around both pitch circles, and by then drawing involutes similar to those already found in their proper order through these points.

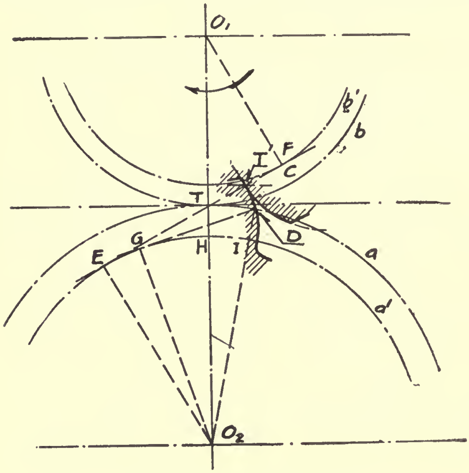


Fig. 258.

In this problem the pitch was made equal to the arcs of approach and recess. However, this need not be the case.

186. Relation Existing between Height of Tooth and Arcs of Approach and Recess.—In a pair of involute gears, Fig. 258, let a and b be the pitch circles; a' and b' the base circles and ETF the line of obliquity of action. Lay down the involutes $T'C$ and $T'D$; then if contact commences at the point T':

$$TC = TD = \text{Arc of approach.}$$

Draw DG normal to the involute $T'D$, then:

$$ET' = GD + \text{arc } EG$$

or since

$$ET = GD,$$

$$TT' = \text{arc } EG,$$

$$\text{triangle } O_2ET = \text{triangle } O_2GD$$

therefore, arc $HI = \text{arc } EG = TT'$.

The application is readily seen in Fig. 259. The pitch circles, base circles, and line of obliquity are the same as those used in Fig. 258.

Lay off TD = arc of approach, and draw DO_2 to intersect the base circle at I. From T mark off TT' = arc HI thus locating point T' where contact begins. A circle through T' having O_2 as a center will be the addendum circle required.

The addendum circle of the driver may be located in a similar manner.

Note that clearance must be allowed between the addendum circle of one gear and the root circle of the other.

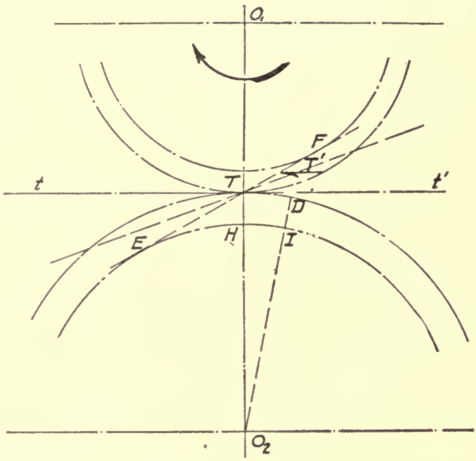


Fig. 259.

187. Example.—Fig. 260, drawn half size, is the practical solution of the following problem, assuming cut teeth.

Distance between shaft centers = 12 inches.
Driving shaft turns at 400 r.p.m.
Driven shaft turns at 200 r.p. m.
Diametral pitch = 4.
Angle of obliquity = $14\frac{1}{2}°$.

$$r_1 = \frac{400}{200+400} \times 12 = 8 \text{ inches radius.}$$

$$r_2 = \frac{200}{200+400} \times 12 = 4 \text{ inches radius.}$$

Number of teeth in gear = $4 \times 16 = 64$.
Number of teeth in pinion = $4 \times 8 = 32$.

Circular pitch = $\frac{\pi}{4} = 0.7854$ inch.

From Art. 198:

$$\text{Addendum} = 0.3183\ p' = 0.3183 \times 0.7854 = 0.25.$$
$$\text{Dedendum} = 0.3683\ p' = 0.3683 \times 0.7854 = 0.289.$$
$$\text{Radius of addendum circle (gear)} \quad = 8 + 0.25 = 8.25.$$
$$\text{Radius of addendum circle (pinion)} = 4 + 0.25 = 4.25.$$
$$\text{Radius of dedendum circle (gear)} \quad = 8 - 0.289 = 7.711.$$
$$\text{Radius of dedendum circle (pinion)} = 4 - 0.289 = 3.711.$$

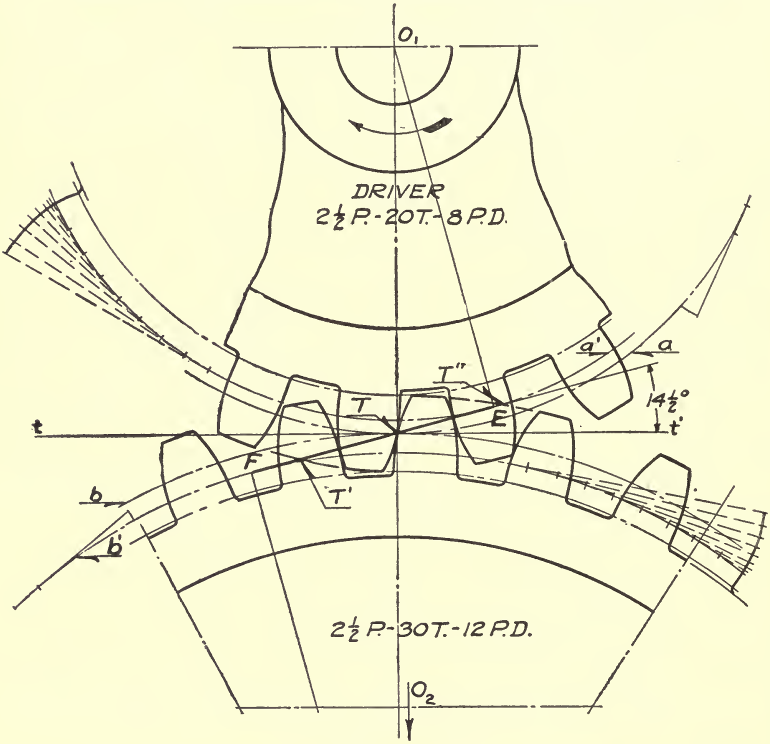


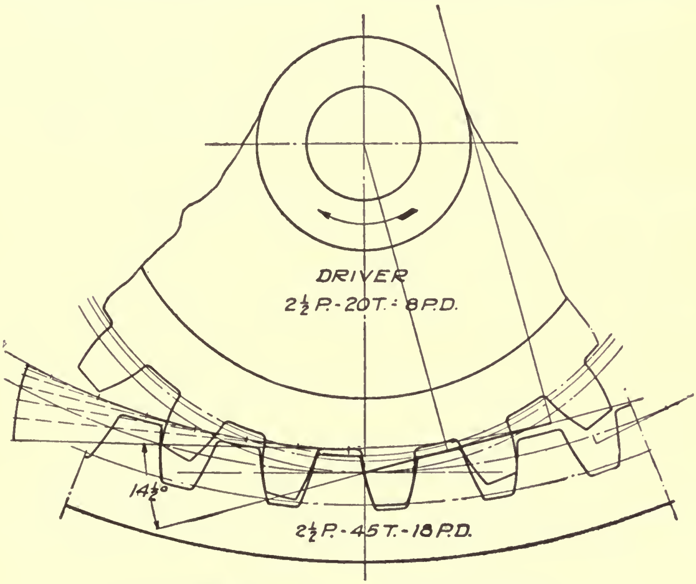
Fig. 260.

Lay down the pitch, addendum and dedendum circles for both gears, making the upper wheel the pinion, and draw the common tangent tTt'. Draw the line of obliquity through T, making an angle of $14\frac{1}{2}°$ with tTt_1. Lay down the perpendiculars O_1E and O_2F and with these lengths as radii draw the base circles a' and b'. The addendum circle with center at O_1 will cut the line of obliquity in the point T' which will, therefore, be the point

where the teeth quit contact. Similarly the addendum circle with center at O_2 will determine the point T'' where contact begins.

Step off divisions equal to one-half the pitch around both pitch circles *a* and *b*, and through the points thus located draw the tooth profiles in their proper order. The tops of the teeth and the bottoms of the tooth spaces are evidently circular arcs.

188. Annular or Internal Gears.—As stated in "the Involute Gear,"[1] "the internal gear is commonly defined as being an



Involute Annular Gear and Pinion.

Fig. 261.

external gear with the teeth turned inside out." Hence the addendum and dedendum are reversed in position and the tooth profile is concave instead of convex, see Fig. 261. This condition changes the tooth action to some extent, so that in special cases it is necessary that the conditions existing be carefully understood. Where the number of teeth on the pinion is small as compared with the number of teeth on the gear, the tooth shape

[1] Published by the Fellows Gear Shaper Co.

requires no special consideration. The Fellows Gear Company propose as a general rule that the smallest difference between the number of teeth in the pinion and the number of teeth in the internal gear with which proper tooth action can be obtained should be 7 teeth for 20° stub-tooth form, and 12 teeth for full length $14\frac{1}{2}°$ involute form.

Several points should be noted relative to annular gears:

(1) The center distance is much shorter than in external gears of the same ratio.

(2) The pitch circles are inclined in the same direction resulting in a greater length of tooth contact. The sliding action between the teeth in contact is reduced on account of the smaller difference in angular velocity.

Fig. 261 shows the layout of an annular gear.

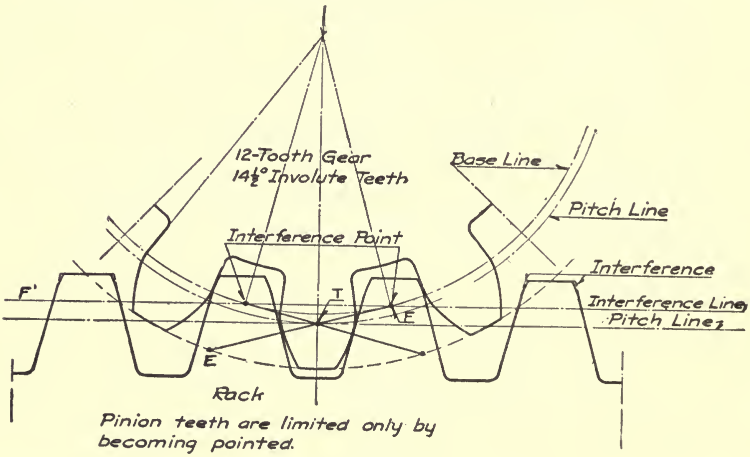


Fig. 262.

189. Rack.—When the pitch circle becomes of an infinite diameter, as in a rack, the base circle will also become of an infinite diameter, and the involute will become a straight line. Hence in a rack the profiles of both face and flank are straight lines and these lines are always perpendicular to the line of obliquity. While the addendum of a rack tooth is limited as explained in

the following article, the addendum of a tooth on the mating gear is limited only by its becoming pointed. Fig. 262 shows the layout of a rack.

190. Interference. Limit of Addendum.—In Fig. 263 wheel *A* has 30 teeth and wheel *B* has 36 teeth, of 3-diametral pitch. The proportions follow the Brown & Sharpe standard for $14\frac{1}{2}°$ involute teeth as listed in Art. 198. Let us assume that it is desired to lengthen the teeth of the wheels. *ETF* is the line of contact and it determines the base circles a' and b'. The teeth of wheel *B* can be lengthened until the addendum circle of the teeth passes through point *E* without causing interference. If carried beyond this line the addendum of each tooth will have to be altered in shape to prevent cutting into the root of the tooth on the other wheel. In the same way the teeth of wheel *A* may be lengthened until the addendum circle passes through point *F*, provided that the teeth do not become pointed before this length is reached.

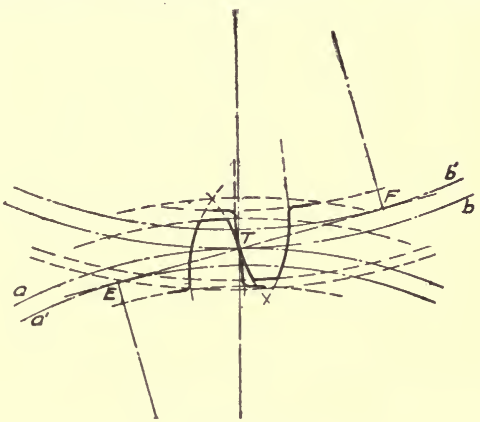


Fig. 263.

Points *E* and *F* are therefore called *interference points* and the addendum lines through these points are called *interference lines.*

In practice it is found that where the difference in the size of the gears is great or in the case of a small pinion and a rack interference is almost certain to occur. The latter case is shown in Fig. 262. If we keep in mind the fact that contact is always along the line *EF* and that it cannot pass beyond these points in either direction it is evident that the teeth of the rack cannot usefully be extended beyond the interference line, *FF'*. The portion between the line *FF'* and the top of the teeth may be altered to clear the teeth of the pinion.

It should be noted that the interference line usually lies out-

side of the addendum line in which case no interference occurs.

191. Approximate Method for Laying-out the Involute Tooth.—In making an approximate lay-out of the involute tooth, Fig. 264, according to the system devised by Mr. G. B. Grant, the pitch, addendum, root, and clearance circles should be laid down. The teeth should then be stepped off on the pitch circle starting at the pitch point. Draw the base line *B* one-sixtieth of the pitch diameter inside the pitch line *A*. Note that all centers are located on the base line *B*. Using the radius *b*, Table III or IV, draw that part of the tooth profile lying between the

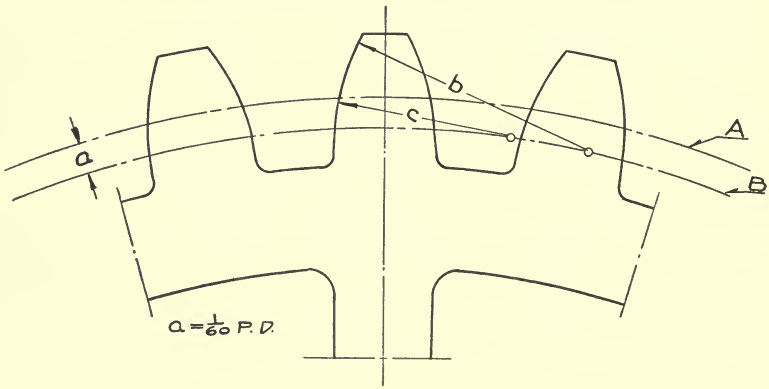


FIG. 264.

pitch circle and the addendum circle. Next, using the radius *c*, draw that part of the tooth profile lying between the pitch circle and the base circle. That part of the profile between the base circle and the fillet at the root of the tooth is a radial line.

NOTE.—

(1) The table is for $14\frac{1}{2}°$ involute teeth.

(2) The values given are for 1-diametral pitch and 1 inch circular pitch.

(3) The tables can be used for annular gears the same as for spur gears, but care must be taken that the tooth of the gear is cut or rounded off to avoid interference.

TABLE III

Number of Teeth	Divide by the Diametral Pitch		Number of Teeth	Divide by the Diametral Pitch		Number of Teeth	Divide by the Diametral Pitch	
	Rad. *b*	Rad. *c*		Rad. *b*	Rad. *c*		Rad. *b*	Rad. *c*
10	2.28	0.69	22	3.49	2.06	34	4.33	3.09
11	2.40	0.83	23	3.57	2.15	35	4.39	3.16
12	2.51	0.96	24	3.64	2.24	36	4.45	3.23
13	2.62	1.09	25	3.71	2.33	37–40	4.20	
14	2.72	1.22	26	3.78	2.42	41–45	4.63	
15	2.82	1.34	27	3.85	2.50	46–51	5.06	
16	2.92	1.46	28	3.92	2.59	52–60	5.74	
17	3.00	1.58	29	3.99	2.67	61–70	6.52	
18	3.12	1.69	30	4.06	2.76	71–90	7.72	
19	3.22	1.79	31	4.13	2.85	91–120	9.78	
20	3.32	1.89	32	4.20	2.93	121–180	13.38	
21	3.41	1.98	33	4.27	3.01	181–360	21.62	

TABLE IV

Number of Teeth	Multiply by the Circular Pitch		Number of Teeth	Multiply by the Circular Pitch		Number of Teeth	Multiply by the Circular Pitch	
	Rad. *b*	Rad. *c*		Rad. *b*	Rad. *c*		Rad. *b*	Rad. *c*
10	0.73	0.23	22	1.11	0.66	34	1.38	0.99
11	0.76	0.27	23	1.13	0.69	35	1.39	1.01
12	0.80	0.31	24	1.16	0.71	36	1.41	1.03
13	0.83	0.34	25	1.18	0.74	37–40	1.34	
14	0.87	0.39	26	1.20	0.77	41–45	1.48	
15	0.90	0.43	27	1.23	0.80	46–51	1.61	
16	0.93	0.47	28	1.25	0.82	52–60	1.83	
17	0.96	0.50	29	1.27	0.85	61–70	2.07	
18	0.99	0.54	30	1.29	0.88	71–90	2.46	
19	1.03	0.57	31	1.31	0.91	91–120	3.11	
20	1.06	0.61	32	1.34	0.93	121–180	4.26	
21	1.09	0.63	33	1.36	0.96	181–360	6.88	

The profiles of the rack teeth are straight lines inclined at $14\frac{1}{2}°$ with the vertical. The outer half of the addendum should be drawn by means of a circular arc having its center on the pitch line of the rack and having a radius of 2.10 inches, divided by the diametral pitch or 0.67 times the circular pitch.

192. Stub-tooth Gear.—While the standard tooth of to-day is the involute having a $14\frac{1}{2}°$ angle of obliquity, its use is not as universal, as might be supposed. A form of tooth known as the "Stub-tooth" has been successfully applied to automobile drives, machine tools, hoisting machinery, etc., and has become established to a degree that many do not realize.

Stub teeth are made on the involute system, the features of this form of tooth being a 20° angle of obliquity (usually), and a shorter addendum and dedendum than used for ordinary gears.

The minimum length of the tooth must be such that continuous action is obtained when using the smallest gear of the set, usually having twelve teeth. A tooth longer than required results in undue friction and wear.

The dimensions of stub-tooth gearing as made by the Fellows Gear Shaper Company are given in Table V. These dimensions are based on two diametral pitches. The clearance is made greater than in the ordinary gear tooth system and equals 0.25 divided by the diametral pitch.

TABLE V

Pitch	Thickness on the Pitch Line	Addendum	Dedendum
4/5	0.3925	0.2000	0.2500
5/7	0.3180	0.1429	0.1785
6/8	0.2617	0.1250	0.1562
7/9	0.2243	0.1110	0.1389
8/10	0.1962	0.1000	0.1250
9/11	0.1744	0.0909	0.1137
10/12	0.1570	0.0833	0.1042
12/14	0.1308	0.0714	0.0893

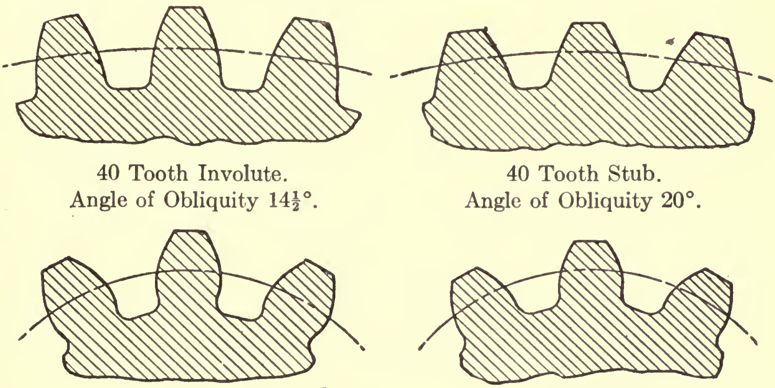
The numerator of the fraction is used in obtaining the dimensions of the thickness of tooth, the number of teeth and the pitch diameter. The denominator is used in obtaining the dimensions

for the addendum and dedendum. As an example, let us choose the 6/8 diametral pitch. Here, the numerator signifies that the gear is of 6-diametral pitch, while the denominator signifies that the addendum is one-eighth of an inch high. A 24-tooth gear of this pitch would have a pitch diameter of 4 inches and an outside diameter of $4\frac{1}{4}$ inches.

193. Stub-tooth Gears. Nuttall System.—Mr. C. H. Logue of the R. D. Nuttall Company has designed a system of stub-gear teeth in which the dimensions are based directly upon the circular pitch.

The dimensions are arrived at as follows:

Addendum $=0.250\times$the circular pitch.
Dedendum $=0.300\times$the circular pitch.
Whole depth of tooth $=0.55\times$the circular pitch.
Working depth of tooth $=0.50\times$the circular pitch.



40 Tooth Involute. Angle of Obliquity $14\frac{1}{2}°$.

40 Tooth Stub. Angle of Obliquity 20°.

12 Tooth Involute. Angle of Obliquity $14\frac{1}{2}°$.

12 Tooth Stub. Angle of Obliquity 20°.

FIG. 265.

194. Comparison between $14\frac{1}{2}°$ Involute and Stub-tooth Gears.—The advantages of the stub-tooth gear may be stated as follows:

(1) Greater strength. Fig. 265 gives a clear idea of the increase of strength with the increase in the angle of obliquity.

(2) Extreme sliding avoided. At the pitch point there is pure rolling contact. At other positions the slid-

ing is proportional to the distance from the point of contact to the pitch point.

(3) More even wearing contact.

(4) Interference is a less serious problem in stub teeth.

195. Interchangeable Wheels.—For interchangeable wheels in the involute system the only conditions that have to be satisfied are a common pitch and the same angle of obliquity for all wheels.

196. Comparison of Systems.—In the involute system the distance between centers may be either greater or less than the theoretical distance without affecting the velocity ratio. This is an advantage not possessed by cycloidal gears. Therefore two wheels of different numbers of teeth, turning about one axis, can be made to gear correctly with one wheel. This feature can be utilized in making differential motions of various kinds.

The profile of the involute tooth is a single curve, while the cycloidal profile is a double curve. Therefore the involute tooth is easier to form.

In the involute system the angle of obliquity is constant and therefore the pressure tending to separate the gears is constant. Hence, the wear is more uniform.

In the cycloidal system a convex surface is always in contact with one that is concave. Although theoretically there is line contact, practically there is surface contact, hence the wear is not so rapid as in involute teeth.

Interference is greater in involute teeth.

197. Proportions of Cast Teeth.[1]—The proportions of cast gear teeth are not standardized. The following proportions in terms of the circular pitch have proven satisfactory in actual practice:

Pressure angle or angle of obliquity $= 15°$.
Length of the addendum . . . . $= 0.3p'$.
Length of dedendum . . . . $= 0.4p'$.
Whole depth of the tooth . . . . $= 0.7p'$.
Working depth of the tooth . . . . $= 0.6p'$.
Clearance of the tooth . . . . $= 0.1p'$.
Width of the tooth space . . . . $= 0.525p'$.
Thickness of the tooth . . . . $= 0.475p'$.
Backlash . . . . $= 0.05p'$.

[1] Leutwiler, Machine Design.

198. Proportions of Cut Teeth.—The formulas proposed by the Brown & Sharpe Co. are used more extensively than any others and are commonly recognized as the standard for cut gear teeth. However, the other systems included in Table VI are used sufficiently to warrant their inclusion. The formulas due to Hunt apply to short teeth, while those due to Logue apply to the stub-tooth system. Proportions for the Fellows' stub-tooth system are to be found in Table V.

TABLE VI

	Brown & Sharpe	Hunt	Logue
Pressure angle	$14\frac{1}{2}°$	$14\frac{1}{2}°$	20°
Length of addendum	$0.3183p'$	$0.25p'$	$0.25p'$
Length of dedendum	$0.3683p'$	$0.30p'$	$0.30p'$
Whole depth of tooth	$0.6866p'$	$0.55p'$	$0.55p'$
Working depth of tooth	$0.6366p'$	$0.50p'$	$0.50p'$
Clearance	$0.05p'$	$0.05p'$	$0.05p'$
Width of tooth space	$0.50p'$	$0.50p'$	$0.50p'$
Thickness of tooth	$0.50p'$	$0.50p'$	$0.50p'$

199. Bevel Gears.—It has been shown in Art. 162 that the contact surfaces of a pair of bevel friction wheels are frusta of a pair of cones whose vertices are at the point of intersection of the axes. In bevel gearing these surfaces are the pitch cones and teeth are formed on them in a manner analogous to the methods used in spur gearing. It should be noted that:

1. The involute form of tooth is almost universally used for bevel gears.
2. Bevel gears are made in pairs and are not interchangeable.
3. Bevel gears are either cast or cut. The casting process is similar to the casting process for spur gears, but the cutting process is much more difficult because of the changing size and form of the tooth from one end to the other.
4. Bevel gears are divided into three classes:
   (*a*) Plain right-angle bevel gears. Center angle $=90°$. In case the shafts are at right

angles, and the gears have equal pitch diameters, they are known as *Miter Gears*.

(*b*) Acute-angle bevel gears. Center angle less than 90°.

(*c*) Obtuse-angle bevel gears. Center angle more than 90°.

5. The velocity ratios are the same as those of cylindrical gears having the same form of pitch line. See Art. 162.

6. While the pitch surfaces of bevel gears are generally made circular cones, they may be elliptical, internal, or irregular.

200. Form of Teeth.—The correct tooth profile in bevel gearing is the spherical involute *EF*, Fig. 266. Consider the base

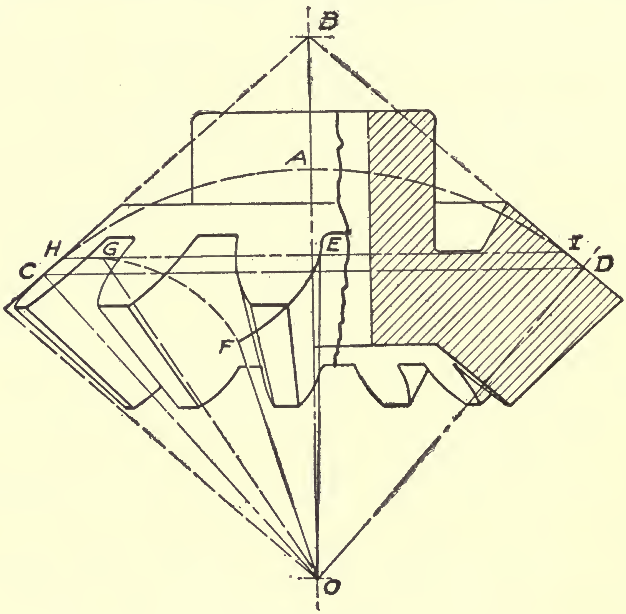


Fig. 266.

cone *OHI* to be enclosed in a thin flexible covering such that the edges meet along the line *OE*. Taking hold of one edge, and keeping the covering taut, unwrap the cover. The involute surface *OEF* is thus generated by the element *OF*. Every point of the

involute *EF* is equidistant from the center *O* and therefore the curve *EF* must lie on the surface of a sphere *HAI*. Hence the curve *EF* is called a spherical involute. The difficulty of laying out tooth curves on a spherical surface is apparent since the surface of the sphere cannot be developed. An approximate method first published by Tredgold, is commonly used. In this method a conical surface *CBD* is substituted for the spherical surface *CAD*. The cone *CBD*, which is called the back cone, is tangent to the sphere at the circle *CD*. As shown in Fig. 266, no appreciable error is introduced by this substitution.

201. Layout of Bevel Gears.—Fig. 267 shows the layout of a pair of right bevel gears, having 18 teeth on the driver and 21

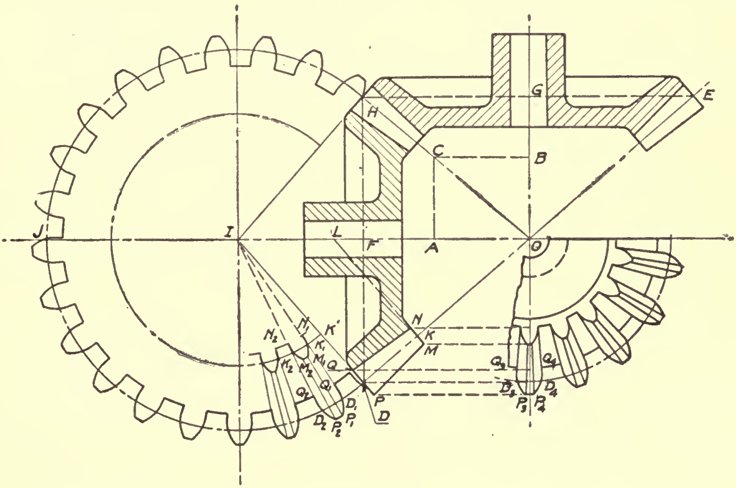


FIG. 267.

teeth on the follower. The velocity ratio is then 7 to 6. The pitch cones and pitch lines are laid out according to the method described in Art. 162. Locate point *I* the vertex of the driver's back cone.

It is evident that the surface which contains the tooth profile has a radius of curvature equal to *IH*. The profile is then laid off on a circle of that radius in precisely the same manner as that used for spur gearing. While the tooth is shown in its true size and shape it is correct for the large end only. It is necessary

to have the profile at the other end of the tooth in order to determine the form for the entire length. Let DK equal the face. Develop the back cone LOK and proceed as before. The addendum and dedendum in this case are the distances KM and KN respectively. Draw P_1M_1, P_2M_2, Q_1N_1, Q_2N_2, D_1K_1, D_2K_2 converging to the center at I. The intersections M_1, K_1, N_1, M_2, K_2 and N_2 will be points on the required profile.

To draw the front view of the gear project points P, D, Q, M, K and N upon the vertical axis through point O. Through the points thus located, draw circular arcs which will be the addendum, root and pitch circles for the profiles. Make $P_3P_4 = P_1P_2$; $D_3D_4 = D_1D_2$, etc., and complete the profiles. The remaining teeth should be laid out in a like manner.

202. Transmission between Non-intersecting Shafts.—The transmission of rotary motion between non-parallel, non-intersecting shafts has been accomplished in four ways:

$$\left.\begin{array}{l}\text{1. Spiral or Helical Gears}\\ \text{2. Worm Gears}\end{array}\right\}\text{Screw Gearing.}$$

3. Skew Bevel Gears.
4. If the shafts are located far enough apart two pairs of ordinary bevel gears may be installed.

203. Screw Gearing.—Screw gearing is a term applied to all classes of gears in which the teeth are of screw form. The pitch surfaces are cylindrical as in spur gearing, but the teeth are not parallel to the axis. Each tooth winds helically like a screw thread. Two classes arise:

1. *Helical Gears.*—When the number of threads or teeth on the cylinder is increased to such an extent that any one thread does not make a complete turn, the resulting gear is called a helical or spiral gear.
2. *Worm Gears.*—In Fig. 269 cylinder a may be provided with one or more threads. If from one to three are used the cylinder and threads together are called a *worm.*

204. Helical Gears, for Parallel Shafts.—A helical gear is simply a spur gear with the teeth twisted. If a gear should be built up of a number of disks and if after the teeth had been cut,

across all of the disks at one time, they were then slightly twisted relatively to one another, the gear would be made up of a series of steps, or in other words it would be a *stepped gear*. Again if the disks were made thin enough lines joining corresponding points of the stepped tooth, profiles would be helices and the gear would be called a helical gear.

Single helical gears produce an end-thrust in each shaft. This objection is overcome by the use of a double-helical or herring-bone gear.

The profiles of the teeth on the reference plane follow the involute form, stub teeth being most generally used.

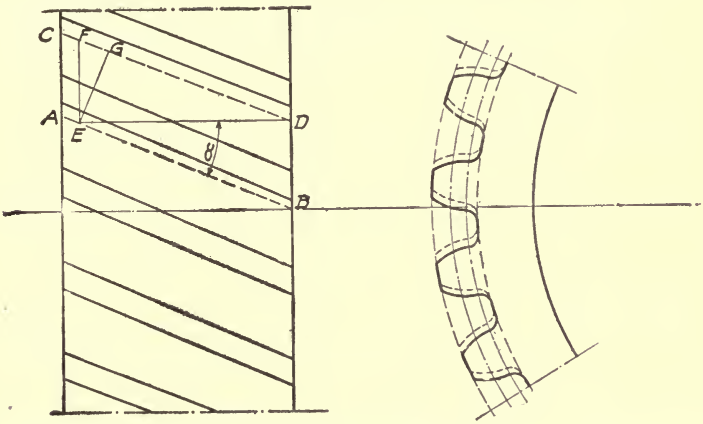


Fig. 268.

Where smoothness of action at high speeds or when a great change in velocity is required, helical gears are usually installed.

The teeth may be either right- or left-handed. In the case of the herringbone or of the Weust gear,—the teeth of the Weust gear are staggered—both right- and left-hand teeth are used on the same wheel.

In Fig. 268 let *AB* and *CD* be corresponding helices on the pitch surfaces of two consecutive teeth. Then *EF*, measured around the surface at right angles to the axis, is the circular pitch. The distance *EG*, measured at right angles to the teeth, is called the normal pitch. The distance *ED*, measured parallel to the axis, is called the axial pitch. The angle α denotes the helix

angle, that is, the angle between the helix and a line parallel to the axis. In mating gears the normal pitch and circular pitch must be the same. The velocity ratio follows exactly the same laws as for ordinary spur gears.

205. Advantages of Double Helical Gears.—When compared with spur gears helical gears have the following advantages:

(*a*) The tooth comes into contact gradually, the action beginning at one end and working over to the other end. In spur gearing the contact takes place across the entire face of the tooth at once.

(*b*) The face of the gear is made long enough so that more than one tooth is always in action. This gives a continuity of action that is in no wise dependent upon the number of teeth in the pinion as in spur gearing.

(*c*) Due to the continuity of action and the nature of the contact the load is transferred from one tooth to another without shock. This also gives smoothness of motion and equalizes the wear, hence the tooth profile is not altered.

(*d*) High gear ratios may be used.

(*e*) All of the advantages of single helical gears are retained. In addition to this double helical gears eliminate end-thrust.

(*f*) Since vibration and noise are eliminated, double helical gears may be run at high pitch line speeds.

206. Helical Gears for Non-parallel, Non-intersecting Shafts.—As noted in Art. 202, helical gears are used to transmit motion between non-parallel, non-intersecting shafts as illustrated in Fig. 269. When thus used the gears give point contact only.

In mating gears the normal pitches, but not necessarily the circular pitches, must be the same. The radii of the pitch cylinders have no direct influence on the velocity ratio, the r.p.m. of the shafts being inversely as the number of teeth.

Let:

N_a and N_b = Speed of shafts AA and BB in r.p.m.

n_a and n_b = Number of teeth in wheels AA and BB respectively.

r_a and r_b = Radii of pitch cylinders AA and BB respectively.
α = Helix angle of wheel AA.
β = Helix angle of wheel BB.
θ = Angle between shafts.
d = Shortest distance between shafts.
p_c = Circular pitch.
p_n = Normal pitch.

Then referring to Fig. 268:

$$p_n = p_c \cos \alpha.$$

$$n = \frac{2\pi r}{p_c} = \frac{2\pi r \cos \alpha}{p_n}.$$

Referring to the two wheels shown in Fig. 263 we have:

$$n_a = \frac{2\pi r_a \cos \alpha}{p_n}.$$

$$n_b = \frac{2\pi r_b \cos \beta}{p_n}.$$

From which:

$$\frac{n_a}{n_b} = \frac{r_a \cos \alpha}{r_b \cos \beta} = \frac{N_b}{N_a}.$$

Also:

$$r_a + r_b = d.$$

207. Example.—The following data apply to a pair of helical gears having pitch cylinders similar to those shown in Fig. 269.

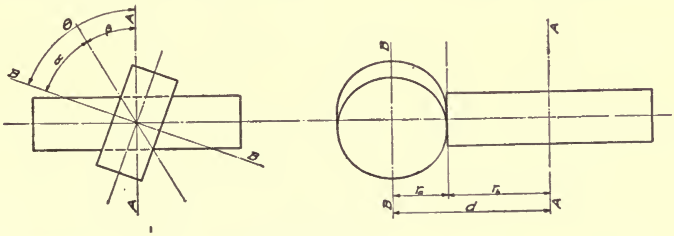


FIG. 269.

Angle between shafts . . . = 70°.
Velocity ratio . . . . = 3 to 1.
Distance between shafts = 18 inches.

Two methods are open for attacking the problem:

(*a*) We can assume the angle of helix for each cylinder and solve for the pitch radii.

(*b*) We can assume the pitch radii and solve for the respective angles of helix.

The first method is the one usually followed.

It is at once apparent that any number of solutions may be obtained since the shaft angle of 70° can be split into an infinite number of combinations.

First solution:

Assume $\alpha=\beta=35°$.

From the statement of the problem $\frac{N_a}{N_b}=3.$

Then:

$$\frac{N_a}{N_a}=3=\frac{r_b \cos 35°}{r_a \cos 35°}=\frac{r_b}{r_a} \quad \text{or} \quad r_b=3r_a.$$

But:

$$r_a\, r_b=18 \text{ inches.}$$

Therefore

$$r_a=4\tfrac{1}{2} \text{ inches; } r_b=13\tfrac{1}{2} \text{ inches.}$$

Second solution: Let

$$\alpha=30°;\ \beta=40°.$$

Then:

$$\frac{N_a}{N_b}=3=\frac{r_b \cos 40°}{r_a \cos 35°}=0.8846\,\frac{r_b}{r_a} \text{ or } \frac{r_b}{r_a}=\frac{3}{0.8846}=3.39.$$

As before

$$r_a+r_b=18 \text{ inches.}$$

Therefore

$$r_a=4.1 \text{ inches; } r_b=13.9 \text{ inches.}$$

208. Worm Gearing.—When it is desired to obtain high-speed reductions between non-intersecting shafts making an angle of 90° with each other, worm gearing is commonly employed. Figs. 270 and 271 show the two classes of worm gearing in common use.

The worm is simply a special form of helical gear in which the angle of helix is so large that the tooth becomes a thread winding entirely around the pitch cylinder.

Each tooth curve of the worm wheel is a short portion of a helix, the angle of which is the complement of that of the worm.

In a worm and wheel combination the worm is usually the driving element. When it is desired to have the wheel drive the worm it is necessary that the angle of the helix of the wheel be made greater than the angle of repose of the materials in contact. This property of the mechanism is used as a locking device in some elevators and other classes of machinery in that it prevents the machine from running backward.

209. Circular Pitch, Axial Pitch, Pitch.—Since worms were formerly universally cut in lathes the circular pitch system is used in worm and wheel calculations.

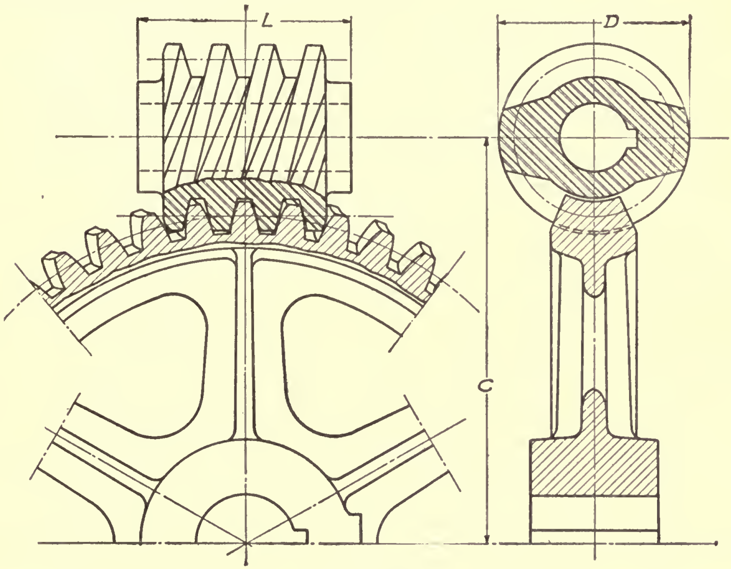


FIG. 270.

The pitch of worm threads, when measured parallel to the axis of the worm, is called *Axial Pitch.* This, of course, must equal the circular pitch of the worm wheel. When the worm is straight or of a cylindrical shape, Fig. 270, the axial pitch is constant for all points of the threads. In the case of the *Hindley worm gear,* Fig. 271, the worm is made smaller at its center than at the ends, and the axial pitch varies for every point, since the angle of helix changes constantly throughout the length of the worm.

The term *pitch* refers to the distance between corresponding points on adjacent threads, measured in an axial direction.

If measured at right angles to the threads it is called the *normal pitch.*

The terms *two-pitch* and *three-pitch* when used in connection with worm gearing refer to the number of threads per inch measured axially on the worm.

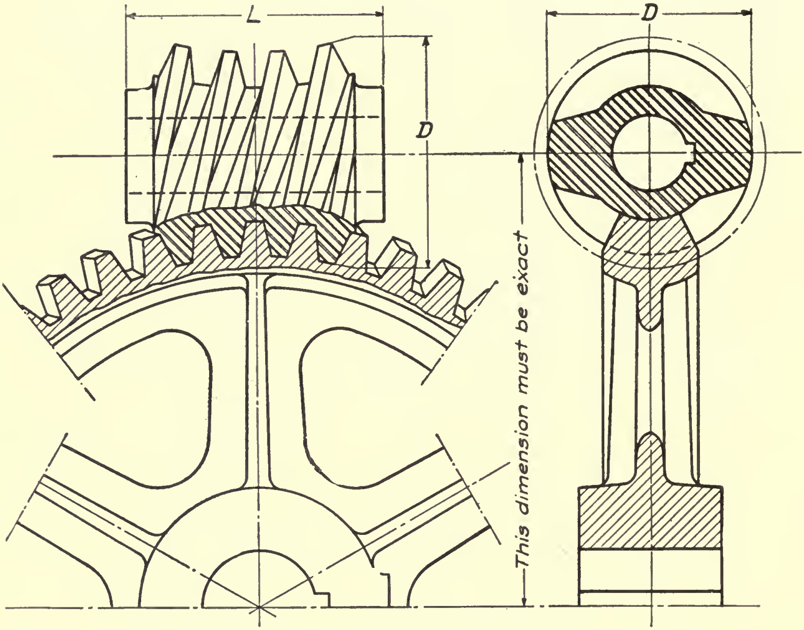


Fig. 271.

Lead refers to the axial distance traversed by a point in one complete revolution of the worm.

210. Velocity Ratio.—In worm gearing the velocity ratio does not depend upon the diameter of the gears, but is the ratio between the number of threads on the worm and the number of teeth on the worm wheel. Thus, for a *single thread* worm the velocity ration of the worm to the wheel is the reciprocal of the number of teeth on the worm wheel. For example, if a wheel has 36 teeth it would require 36 turns of the worm to turn the wheel once and the velocity ratio would be $\frac{1}{36}$.

For two-thread and three-thread worms the velocity ratio would be $\frac{2}{36}=\frac{1}{18}$ and $\frac{3}{36}=\frac{1}{12}$ respectively.

Thirty-one teeth, with few exceptions, is the least number of teeth that it is advisable to use on a worm wheel, in order to avoid undercutting. Hence, if the velocity ratio is less than thirty the worm must be provided with more than one thread. For example, let the worm shaft make 18 revolutions while the wheel shaft turns once. Two solutions are possible:

Teeth on worm wheel.............	36 : 54
Teeth on worm....................	2 : 3

211. Comparison of Hindley and Cylindrical Worm Gears.—The efficiency and load carrying capacity of the Hindley worm gear is slightly better than that of the cylindrical worm gear. However, the use of the Hindley gear presents the following restrictions:

(1) The center distance of the worm and gear must be exact.

(2) The worm axis must be in proper alignment with the gear.

(3) The plane of the gorge circle of the worm must be located directly over the center of the wheel.

Cylindrical worm gears are often made longer than is required. This construction allows the worm to be moved along the shaft, thus presenting a new surface in the event of wear.

It should be noted, Fig. 270, that the teeth of the worm on the central section are those of an involute rack, while those of the wheel are of the involute form. The pressure angle used in laying out the teeth is generally $14\frac{1}{2}°$.

212. Skew Bevel Gears.—When the distance between the shafts is small skew bevel gears alone will answer the purpose.

Nearly all of these gears are cast from patterns, the cut gears of this type not being even good approximations of the correct forms.

No form of tooth has yet been found that can be cut and still possess the qualities of strength, reversibility and low pressure angle.

The three main systems so far attempted are:

(*a*) *Willis Epicycloidal System.*—The tooth profiles are here generated by a hyperboloid rolling inside of one and outside of the other pitch hyperboloid. Prof. MacCord has shown that teeth so generated are not tangent in all positions.

(*b*) *Olivier Involute System.*—The tooth profiles are here helical convolutes. It is difficult to find a satisfactory position for the teeth since they vanish at the gorge and since the obliquity of action increases rapidly as we leave the gorge.

(*c*) *Beale Skew Gear.*—This is a modification of the Olivier system in which the teeth do not vanish at the gorge. The teeth are undercut and are hard to make.

213. Comparison of Systems.

(*a*) *Spiral or Helical Gears.*—Gears of this type work satisfactorily if the center distance is slightly altered or if either of the gears is shifted slightly along the shaft. However, they wear rapidly because of point contact.

(*b*) *Worm Gears.*—A worm mating with a straight-faced helical gear has the same advantages and disadvantages as spiral gears. A straight worm mating with a hobbed wheel gives line contact. In this case, however, the shafts and gears must be definitely fixed in position.

(*c*) *Skew Bevel Gears.*—These gears have straight teeth and give line contact. They are little used, however, since it is very difficult to produce correctly shaped teeth.

(*d*) If the shafts are located far enough apart two pairs of ordinary bevel gears may be installed.

CHAPTER XI

CAMS

214. General Principles.—A *cam* is a machine element, so shaped that, by its oscillating or rotating motion, it gives a predetermined motion to another link called the follower.

It should be noted that:

(1) The *cam* is usually the driving link.

(2) The *velocity* of the cam is generally *uniform.*

(3) The motion of the *follower* is either *reciprocating* or *angular* about a fixed axis. In either case the follower usually moves in a non-uniform, irregular manner.

(4) *Equal movements* of the *cam* are not usually accompanied by equal movements of the *follower.*

(5) *Line contact* is almost always present between the cam and follower.

(6) Cams are used to produce motions *not easily gotten* by other means.

(7) To decrease *wear,* the irregularity is usually confined to the cam and the follower is provided with a *roller* whenever possible.

(8) Cams in which the follower is compelled to follow a fixed predetermined path due to the nature of the cam groove are called *positive motion cams.*

When springs are used to return the follower to its original position they must be strong enough to perform their work properly. In this case the effort necessary to compress or extend the spring must be added to the effort necessary to do the work.

In some cases springs are necessary but positive motion cams are to be preferred.

(9) *Rotary cams* are divided into two general classes:

(*a*) Cams in which the only requirement is that the movement of the follower shall begin at a certain position of the cam shaft and that the movement of the follower shall end at a second position of the cam shaft, the follower having moved a certain distance.

(*b*) Cams in which the follower shall at all times occupy predetermined positions.

Cams of class (*a*) are far more numerous and they allow greater liberty in the design of the cam groove.

Cam grooves of class (*b*) can be altered somewhat by increasing the diameter of the cam.

215. Layout of Cams.—The first step in designing a cam is to determine the required movement of the follower at the point where the work is to be done in order that the given operation may be performed.

The movement thus determined should be plotted—full size where possible—as shown in Fig. 272. The height of the chart

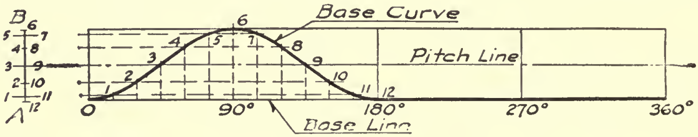


Fig. 272.

should be the same as the total motion of the follower. Since the cam chart is the development of the cam equal divisions on the *pitch line* represent equal angular displacements of the cam. Through the points of division thus determined vertical lines are laid down on which are laid off from the *base line* the total movement for the chosen position. The divisions thus located are numbered and the same measurement and number is placed on the left-hand vertical for use in the cam layout, as shown by the line *AB* in Fig. 272.

The irregular line joining the points, thus located, is known as the *base curve.* It will be seen that the rising and falling of the base curve relative to the base line represents the movement

of the cam roller or follower. Hence, in the cam the base curve plays a part analogous to that of the pitch surface in gears. The angle which the base curve makes with the base line is known as the *pressure angle.* It may be noted that the velocity of the follower is proportional to the *tangent* of this angle. Wherever possible, the maximum pressure angle of the set on the working stroke should not exceed 30°. The base curve thus determines the motion of the follower and also the pressure angle.

In the preceding discussion but one cam was considered. In many machines several cams, whose functions are dependent on each other, are used. When this is the case the base curves of all cams are usually plotted on the same chart, which is called a *timing diagram.* These diagrams are of value in determining the proper sequence of events in the mechanism and also in providing against possible interference. All cams of a set usually have the same diameter, since this facilitates the layout.

216. Base Curves.—Where the only requirement is that the movement of the follower shall begin at a certain position of the cam shaft and that the movement of the follower shall end at a second position of the cam shaft, the follower having moved a certain distance, the choice of a suitable base curve is left to the designer.

The following base curves are in common use:

(*a*) *Straight Line.*—This base curve gives an abrupt starting and stopping velocity, which produces a considerable shock in the follower. After starting, the velocity of the follower is constant, hence the acceleration, while theoretically infinite at the start and end of the stroke, is zero during the stroke.

(*b*) *Straight-line Curve Combination.*—This base curve is similar to the straight line with the exception that the ends are rounded off. The addition of the curve does away with the actual shock of the straight line, but it gives a very sudden action. The straight line and straight-line curve combinations are objectionable, and their use should be avoided.

(*c*) *Crank or Harmonic Motion Curve.*—The construction of this curve is shown in Fig. 273. *AB* is the total motion of the follower and A_1A_2 is the developed

circumference of the pitch circle of the cam between two chosen positions of the cam shaft. Draw a semicircle on the line CD and divide the semicircumference into any number of equal parts. Divide the line A_1A_2 into the same number of equal divisions and at the points thus located erect perpendiculars. These parts will represent equal intervals of angular displacement of the cam, and if the cam is moving with uniform angular velocity these parts will also represent equal intervals of time. The construction is shown in the figure. This curve gives harmonic motion to the follower, that is, the velocity follows the law $V = c\omega \sin \omega t$ where c is one-half the travel of the follower. The acceler-

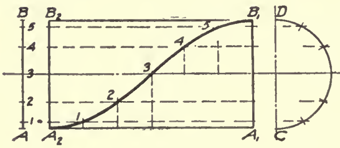


FIG. 273.

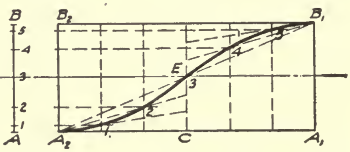


FIG. 274.

ation is zero at the middle of the stroke. On either side of the center the acceleration increases according to the law $A = c\omega^2 \cos \omega t$.

(*d*) *Parabola.*—Fig. 274 shows the case where the follower is to rise half the distance AB with uniform positive acceleration, and to complete its travel to B with uniform retardation, that is, with uniform negative acceleration. The curve is made up of parts of two parabolas, A_2E and EB_1. The usual and most convenient construction for drawing the parabola is shown in the figure. In this construction, the line EC is divided into any convenient number of equal parts and each of the division points is connected by a straight line to A_2. The line A_2C is divided into the same number of equal parts and at each division point a perpendicular is erected. The intersections 1, 2, 3, etc., are points on the required curve.

(*e*) *Elliptical Curve.*—The method of constructing this curve is similar in every respect to that used for the harmonic motion curve, the only difference being the substitution of an ellipse for the semicircle shown in Fig. 273. The single advantage of this curve is that it gives slower starting and stopping velocities to the follower. The velocity at the center of the stroke is higher and the acceleration is variable, reaching its maximum at the ends of the stroke.

Mr. W. B. Yates recommends a ratio of the vertical axis of the ellipse to the horizontal axis of 1 to $1\frac{3}{4}$. If this ratio be increased the follower will start and stop with less acceleration while the velocity at the center will be considerably higher.

217. Maximum Pressure Angle Factors.—Fig. 275 shows a layout of the base curves in common use, plotted for a maximum slope of 30°.

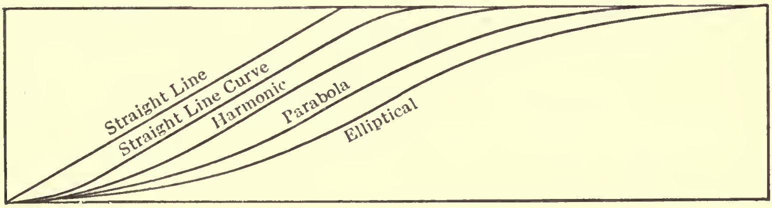


Fig. 275.

AB represents the total motion of the follower, and this length is taken as unity for comparison. It will be noted that the length of the base curves vary widely. For example, take the case of the crank curve for a maximum pressure angle of 30°. Here the length required is equal to 2.72 times the throw or travel of the follower. In the case of the elliptical curve for the same maximum pressure angle the length required is 3.95.

Table I [1] and the accompanying chart [1] gives these values for various pressure angles and for various types of curves.

[1] From "Cams" published by John Wiley & Sons, Inc. Printed by permission of Franklin De. R. Furman.

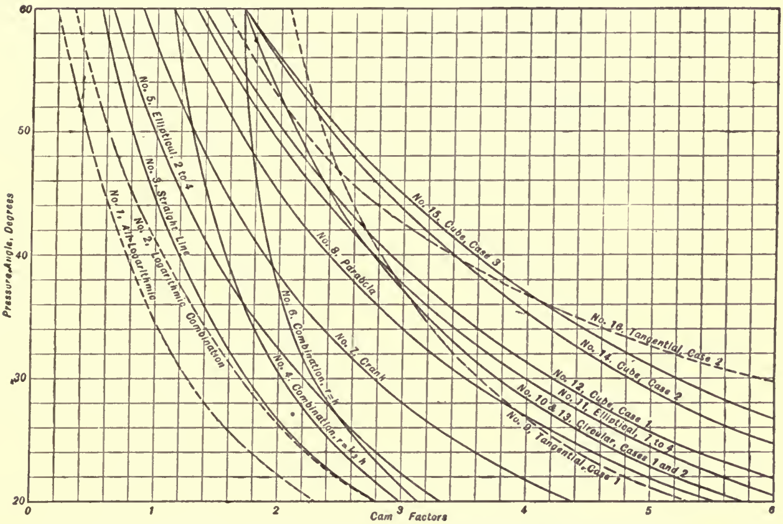
TABLE I

Table of Pressure Angle Factors

No.	Name of Base Curve	Factor for Maximum Pressure Angle				
		20°	30°	40°	50°	60°
1	All logarithmic	No general factors				
2	Logarithmic combination	No general factors				
3	Straight line.	2.75	1.73	1.19	0.84	0.58
4	Straight line-curve combination (Radius equal to ½ followers motion)	2.92	2.00	1.56	1.31	1.16
5	Elliptical curve. (Ratios of semi-axes 2 to 4)	3.32	2.17	1.45	1.00	0.68
6	Straight line-curve combination. (Radius equal to followers motion)	3.10	2.27	....	1.77	1.73
7	Crank curve	4.43	2.72	1.87	1.32	0.91
8	Parabola	5.50	3.46	2.38	1.68	1.15
9	Tangential curve Case 1. (Length of straight surface not specified)	No general factors				
10	Circular curve Case 1. (Symmetrical circular arcs)	5.67	3.73	2.75	2.14	1.73
11	Elliptical curve. (Ratio of semi-axes 7 : 4)	6.25	3.95	2.75	1.95	1.35
12	Cube curve Case 1. (Unsymmetrical cube curve)	6.68	4.20	2.90	2.04	1.40
		4.13+ 2.55	2.60+ 1.60	1.79+ 1.11	1.26+ 0.78	0.87+ 0.53
13	Cube curve Case 3. (Symmetrical cube curves)	8.22	5.20	3.56	2.52	1.73
14	Circular curve Case 2. (Unsymmetrical circular arcs)	5.67	3.73	2.75	2.14	1.73
		4.26+ 1.41	2.80+ 0.93	2.06+ 0.69	1.60+ 0.54	1.30+ 0.43
15	Cube curve Case 2. (Cube curve and circular arc)	7.60	4.83	3.37	2.42	1.73
		6.18+ 1.42	3.90+ 0.93	2.68+ 0.69	1.89+ 0.53	1.30+ 0.43
16	Tangential curve Case 2. (Length of straight surface specified)	No general factors				

218. Size of Cams.—If a table similar to Table I is not available, a tedious cut and try method must be employed to determine the length of the base line. By the use of Table I the length of the base line can be determined as follows:

f = Cam factor as given in Table I;
t = Total motion or throw of follower;
n = Degrees of rotation of the cam;
r = Radius of pitch circle of cam.



As an example, consider the crank curve, Fig. 273, and assume a maximum allowable pressure angle equal to 30°. Since the throw AB is always known, multiply this value by the cam factor f, in this case 2.72, to obtain the length B_1B_2 which, in turn, is known as a fraction of the entire circumference, equal to $\frac{(360)}{n}$.

Hence:

$$\text{Circumference} = f \times t \times \frac{360}{n},$$

or

$$C = 360\frac{ft}{n},$$

$$r = \frac{360\,ft}{2\pi n} = 57.3\frac{ft}{n}.$$

219. Rotary Cams. Radial Follower.—Many cams of this type are laid out with entire disregard for technical design. While satisfactory results may often be obtained by experienced designers, a great objection remains, that is, there is no control of the velocity and acceleration of the follower rod.

Fig. 277 shows the theoretical lay-out of a one-step rotary cam having a radial follower and conforming to the following specifications:

Cam to be based on the harmonic motion base curve.

Maximum pressure angle $= 30°$.

The follower shall move:

Up 4 units in 90°.
Rest through 45°.
Down 4 units in 90°.
Rest through 135°.

Solution:

From Table I, $f = 2.72$ per unit rise.
Rise $= 4$ units.

Therefore since the rise takes place in 90°.

Circumference $= 4 \times 2.72 \times \dfrac{360}{90} = 43.52$ units $=$ length of chart.

Radius of pitch circle $= \dfrac{43.52}{2\pi} = 6.93$ units.

Fig. 276 shows the lay-out of the cam chart.

Lay-out of cam.

With O, Fig. 277, as a center and a radius of 6.93 units draw the pitch circle $PCDE$ and divide the area into divisions as follows: $PC = 90°$; $CD = 45°$; $DE = 90°$; and $EP = 135°$. Divide angle POC into 6 equal angles by drawing radial lines. Lay off APB equal to APB, Fig. 276, and through the division points 1, 2, 3, etc., projected on $A'P'B'$ draw circular arcs having O as a center. The intersections of these arcs with the corresponding radial lines give points on the required cam surface.

The maximum angle occurs at the point where the pitch circle cuts the cam surface in practically all cases. This pressure angle

is the angle between a normal and a radial line drawn at the point under consideration. In this case the maximum pressure angle is at $3'$ and is equal to $30°$.

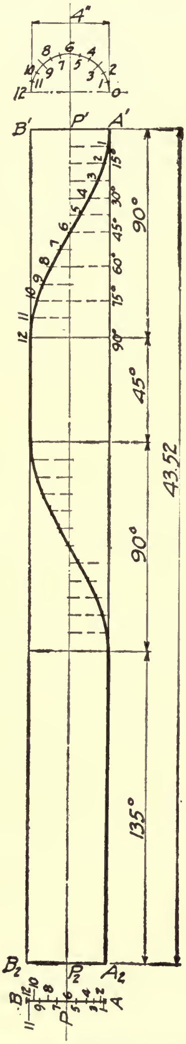


Fig. 276.

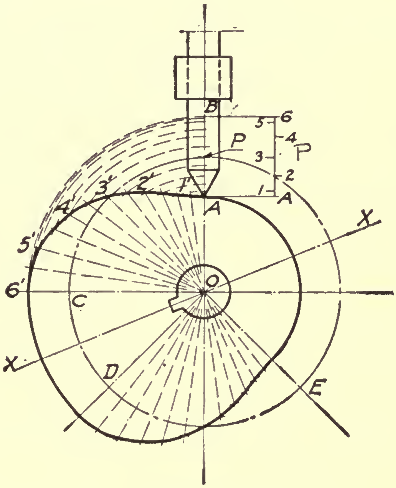


Fig. 277.

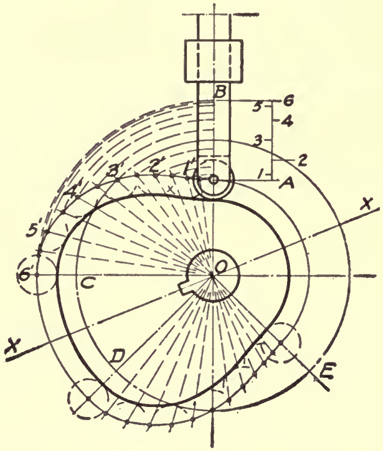


Fig. 278.

220. Working Surface of Cams.—If the follower is fitted with a roller instead of the wedge, as shown in Fig. 278, then point A, Fig. 277, becomes the center of the roller shaft. Hence the work-

ing surface of a cam is always smaller than the pitch surface, except in the case of the wedge follower, when the working surface and pitch surface coincide. The working surface is gotten by drawing circular arcs with consecutive points of the pitch surface as centers, the radius of the arcs being equal to the radius of the roller. The working surface is then the envelope to the circular arcs. Fig. 278, derived from Fig. 277, shows the construction.

221. Two-step Cam.—In the case of a two-step cam, or of a cam having unequal pressure angles on the two strokes, the pitch circle used should always be the larger of the two calculated. For this reason the maximum pressure angle will equal that specified on one step and will be smaller than specified on the other.

As an example, let it be required to determine the diameter of the pitch circle and the pressure angles for a one-step radial cam whose specifications follow:

Cam to be based on the parabolic base curve.

The follower shall move:

Up 3 units in 120°. Max. pressure angle not to exceed 20°.
Down 3 units in 90°. Max. pressure angle not to exceed 40°.
Rest through 150°.

Required radius of pitch circle for working stroke

$$=\frac{360\,ft}{2\pi n}=\frac{360\times 5.5\times 3}{2\times 3.14\times 120}=7.88.$$

Required radius of pitch circle for return stroke

$$=\frac{360\times 2.38\times 3}{2\times 3.14\times 90}=4.545.$$

Hence the pitch circle will have a radius of 7.88 units, and the maximum pressure angle on the working stroke will be 20°.

To find the maximum pressure angle on the return stroke substitute 7.88 for r in the formula, $r=\dfrac{360\,ft}{2\pi n}$, and solve for f which in this case will equal 4.126. Referring to Table I we find this value to correspond to a maximum pressure angle of approximately 26°.

In case of a two-step cam the calculations for each step should be carried out. The calculations should then be compared, using

the longest chart dimension as a standard. In a large number of cases the base chart length will have to be changed in order that no pressure angle shall exceed the maximum specified.

222. Size of Follower Roller.—The radius of the follower roller should always be less than the shortest radius of curvature of the pitch surface. If this rule is followed a smooth curved working surface will result, and the follower will always have the motion for which it was designed.

223. Rotary Cam. Offset Follower.—The method of construction for cams of this type is entirely different from that for

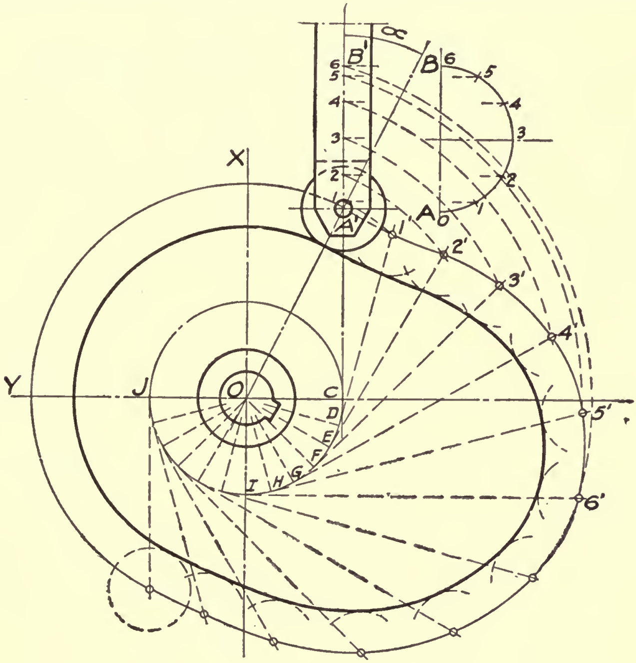


Fig. 279.

cams having a radial follower. The pitch circle here serves no purpose other than that of obtaining a cam of reasonable size and will be omitted in this discussion.

Fig. 279 shows the theoretical lay-out of a one-step rotary cam having an offset follower which moves in a straight line. The specifications follow:

Cam to be based on the harmonic motion base curve.

The follower shall move:

Up 3 units in 90°.
Down 3 units in 90°.
Rest through 180°.

The lowest point of the followers motion to be two units to the right and four units above the cam shaft center.

Solution.—Draw OX parallel and OY perpendicular to the line of the follower's motion. With O as a center and with a radius of 2 units draw the circle C–I–J. The center line of the follower's motion produced will be tangent to this circle at all times. Divide the arc C–I into 6 equal divisions at points C, D, E, etc., and at these points draw tangents to the arc. By the usual construction as shown, locate the divisions on the line AB which is equal to the stroke. The construction for the pitch surface of the cam is completed by making $D1' = C1$; $E2' = C2$; $F3' = C3$, etc., and by drawing a smooth curve through the points A', $1'$, $2'$, etc., thus located.

The working surface is next located by choosing a suitable radius for the roller and using the pitch surface as a line of centers in laying down circular arcs to which the working surface is the envelope. The addition of a suitable hub and keyway completes the cam.

The angle α, Fig. 279, is the pressure angle at the lowest point of the stroke.

224. Rotary Cam. Swinging Follower.—Cams of this type are usually constructed according to one of two systems:

(*a*) Those in which the follower arc if continued would pass through the center of the cam;
(*b*) Those in which the extremities of the follower arc lie on a radial line.

In the first class the pressure angle will be greater on one stroke than on the other, hence cams of this type are used where heavy work is to be done on one stroke only. In cams of the second class the pressure angles on both strokes are nearly equal, hence cams of this type will be chosen where the same amount of work is to be done on both strokes.

Fig. 280 shows the lay-out of a rotary cam having a swinging follower equipped with a roller and conforming to the following specifications:

Follower to be $4\frac{1}{4}$ units long and to move:

Up in 120° following the parabolic base curve.
Rest through 30°.
Down in 90° following the harmonic motion base curve.
Rest through 120°.

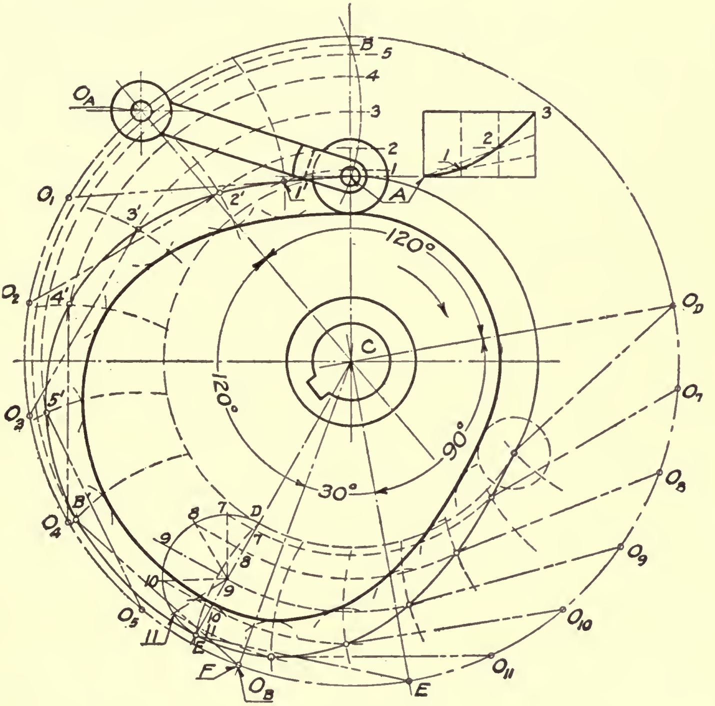


Fig. 280.

The extremities of the follower arc are to lie on a radial line.

The follower is to swing through 35° and the lowest point of its travel is to be $3\frac{7}{8}$ units from the center of the cam shaft.

Solution.—The relative motions of the cam and follower will not be changed if we hold the cam stationary and move the fixed center O_A, of the follower arm in a circle about the center of the

cam shaft C. This procedure will be followed for cams having swinging follower arms.

With a radius of $4\frac{1}{4}$ units strike an arc of 35° and measure the chord. From C lay off CA equal $3\frac{7}{8}$ units and from A lay off AB equal to the length of the chord as determined above. With points A and B as centers and a radius of $4\frac{1}{4}$ units draw arcs intersecting at O_A thus locating the fixed center of the follower arm. With C as a center and a radius CO_A draw a circle which will be the line of centers considering the cam to be fixed and the follower to revolve. Lay down the parabolic curve as shown in the figure. Make the angle O_ACF equal to 120° and divide the arc O_AF into 6 equal divisions at O_A, O_1, O_2, etc. With these points as centers and using a radius equal to O_AA draw the dashed arcs shown. With point C as a center and radii CA, $C1$, $C2$, etc., draw arcs intersecting the dashed arcs, just laid down in points $1'$, $2'$, $3'$, etc., which will be points on the required pitch surface. A similar construction is used for the down stroke, the complete construction being shown in the figure. The working surface is gotten as usual.

While the method just described is not quite correct, it is sufficiently accurate for all practical purposes. The correct method would be to draw the follower in its exact positions and then design the cam to meet the roller at these points.

225. Positive Motion Cams.—Any of the preceding cams could be made into positive motion cams by the addition of a roller where not already in place and a second working surface on the opposite side of the roller. The radius of the roller cannot be made larger than the smallest radius of curvature of the cam grooves.

Cams so generated are called Face Cams. The disks are generally made circular and the groove is cut on one side of the disk only.

226. Rotary Cams. Tangential Follower.—Rotary cams having tangential followers are divided into two general classes:

(*a*) Sliding tangential follower.
(*b*) Pivoted or swinging tangential follower.

The method of procedure in both cases is the same, (*a*) being a special case of (*b*).

Let it be required to lay out a cam that is to operate the slider Y, Fig. 281, by means of the swinging tangential follower X, which is pivoted at O_0, the center of the cam shaft to be located at C.

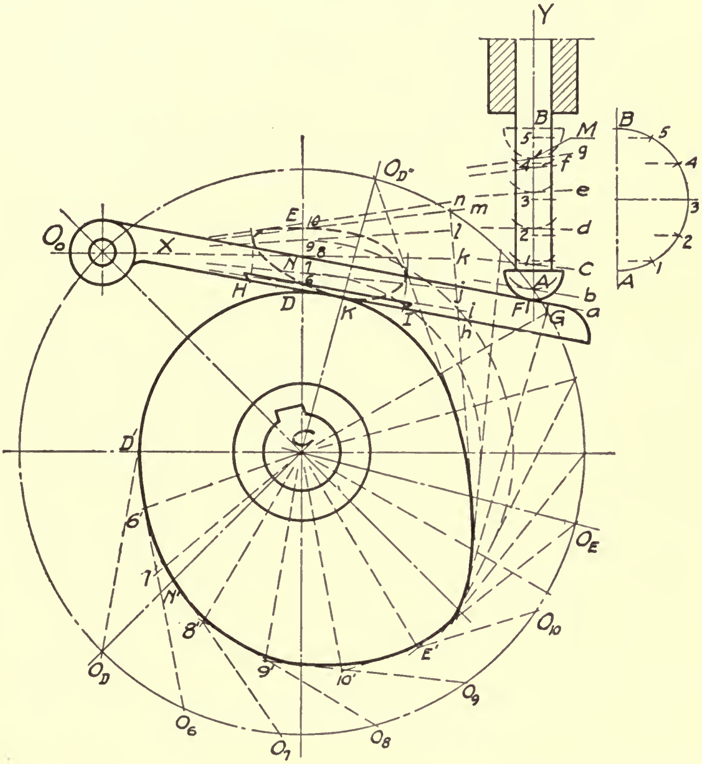


Fig. 281.

The slider Y is to:

Rest while the cam turns through 90°.

Up 3 units with harmonic motion while the cam turns through 120°.

Down 3 units with harmonic motion while the cam turns through 90°.

Rest while the cam turns through 60°.

Solution.—The foot of the slider is a hemisphere with its center at A, which is the lowest position of the slider. Lay off

AB equal to 3 units and divide it into 6 harmonic spaces at *A*, *1*, *2*, *3*, *4*, *5* and *B*. The points thus located will be the successive centers of the hemisphere. Lay out the necessary portions of the arcs as shown, and draw the upper edge of the follower tangent, thus locating lines, *a*, *b*, *c*, . . . *g*. The lines *h*, *i*, *j* . . . *n*, which are the successive positions of the bottom of the follower are next drawn parallel to the upper edges and at a distance below them equal to the thickness of the follower arm. A vertical line through the cam shaft center *C* cuts these lines in the points *D*, *6*, *7*, *8*, *9*, *10* and *E*. Draw the circle of centers having *C* for a center and a radius equal to CO_0.

Lay off the angles:

$$O_0CO_D = DCD' = 90°.$$
$$O_DCO_E = D'CE' = 120°.$$

Then, consider the cam to remain stationary while the remainder of the mechanism revolves counter-clockwise about it. After the rest period of 90° the pivot point of the swinging arm will lie at O_D and the point *D* will lie at *D′*. Hence O_DD' will be a new position of the tangential line of the follower.

To locate other positions, divide the angles O_DCO_E and $D'CE'$ into six equal parts, thus locating the successive centers O_6, O_7, O_8, . . . O_E and the radial lines *C6′*, *C7′*, *C8′* . . . *CE′*. Lay off *C6′* equal to *C6* and join *6′* to O_6, etc. A smooth curve tangent to these lines and joining an arc of radius *CK* will be the working surface of the cam.

The construction for the remainder of the stroke is arrived at in the same manner, the construction being shown in the figure.

The wearing surface between the top of the pivoted follower and the slider is easily determined. With O_0 as a center and a radius to the upper tangential point *M*, draw an arc cutting the upper edge of the follower at *G*. *FG* is then the surface exposed to wear.

To determine the wearing surface between the follower and the cam it is necessary to find the locus of the point of contact, shown by the dashed curve, Fig. 281. Points on the curve are located as follows: *N′* is the point of tangency between the follower edge O_77' and the cam. Therefore, lay off O_0N equal to O_7N' and one point is located.

When the locus is completed draw limiting arcs with O_0 as a center thus locating points H and I. HI is then the limit of wear. This part of the follower is often provided with a replaceable shoe as indicated in the figure.

227. Rotary Cams. Sliding Tangential Follower.—As stated in Art. 226, cams of this type are laid out in precisely the same manner as cams having a swinging tangential follower. The only difference lies in the fact that the centers O_0, O_D, O_6, etc., lie at infinity and the lines joining these centers to D, D', 6, etc., must, therefore, in all cases be perpendicular to the radial division lines CD, CD', $C6'$, etc. The cam is drawn tangent to these lines as before.

The layout of a figure to meet these conditions is left to the student.

In all cams having a tangential follower the working surface of the cam must be made tangent to each tangential line in turn, in order that irregular motion may be avoided. This is not always possible where the data are chosen at will.

It is evident that cams of this type must always be convex externally.

228. Plane Sliding Cams. Sliding Follower.—The flat plate AC, Fig. 282, has a reciprocating horizontal motion, in its own plane, between fixed guides. It carries a curved face which works

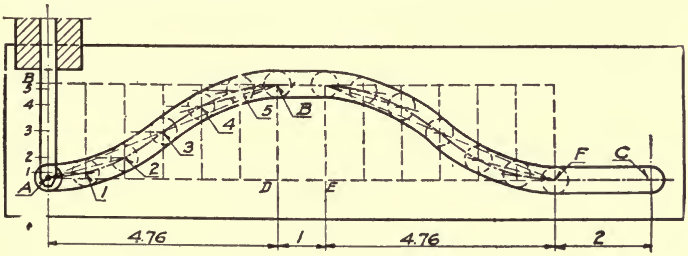


Fig. 282.

in contact with the lower end of the follower AB. The follower is guided in a vertical direction, rising and falling as the cam reciprocates.

Let it be required to lay out a plane horizontal sliding cam having a maximum pressure angle of 40° and working with a sliding follower that is guided in a vertical direction.

The follower to:

Rise 2 units with uniform acceleration, and retardation.
Rest while the cam moves 1 unit.
Down 2 units with uniform acceleration and retardation.
Rest while the cam moves 2 units.

Solution.—The pitch surface of the cam is obtained by assuming that the cam is fixed and that the follower moves toward the right.

The maximum pressure angle factor for a 40° parabolic curve is 2.38. Therefore, the length required for a rise of 2 units is 2 times $2.38 = 4.76$ units.

To any scale, Fig. 282, lay out $AB = 2$ units, $AD = 4.76$ units; $DE = 1$ unit; $EF = 4.76$ units; $FC = 2$ units. Complete the rectangle and lay out the pitch and working surfaces as shown in the figure.

229. Plane Sliding Cam. Pivoted Follower.—Fig. 283 shows the layout of a plane sliding cam having a pivoted follower

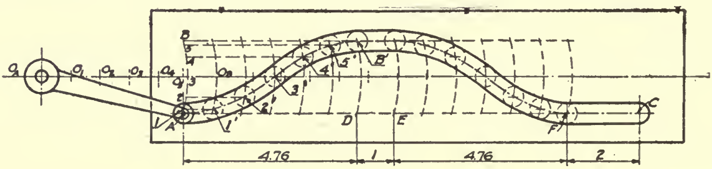


Fig. 283.

4 units long, the condition of the problem being the same as given in Art. 228. The layout of this cam is in all respects similar to the layout of the sliding cam fitted with a sliding follower, the lone difference being that the center of motion of the follower, in the case of the sliding follower it is at infinity, while in the case of the pivoted follower it is at the pivot point.

Solution.—Starting at a point A lay out the distance AB, AD, DE, EF, and FC. With A as a center and a radius of 4 units locate the point O_A, which will be the required pivot center. With O_A as a center and a radius of 4 units strike the arc AB which will be the path of the center of the roller pin. Divide the arc AB into any number of spaces conforming to the parabolic curve (6 being used in this case). Divide the lines AD and EF into the

same number of equal spaces. Through the points of division thus located, draw arcs parallel to the arc AB. These arcs will be the consecutive positions of the path of the center of the roller pin. Through the points of division of the arc AB draw horizontal lines, thus locating points A, $1'$, $2'$, etc., of the pitch surface of the cam. The working surface is located in the usual manner.

230. Cylindrical Cams.—A cylindrical cam may be used to give the same motion as a sliding cam. A cam of this form may be looked upon as a plane sliding cam bent to a cylindrical shape. As the cylinder is rotated on its axis, the follower is given precisely the same motion as when the plane sliding cam is given a motion of translation.

231. Cylindrical Cam. Sliding Follower.—Let it be required to lay out a cylindrical cam, whose follower is to be fitted with a cylindrical roller. The cam is to be designed to meet the following conditions:

Follower to move:

Parallel to the axis of the cam.
Out 2 units in 135° based on Elliptical Curve.
In 2 units in 135° based on Elliptical Curve.
Rest through 90°.
Maximum pressure angle = 40°.

Considering the plane sliding cam to be the development of a cylindrical cam, the length of the pitch cam chart, and the diameter of the pitch cylinder of the cam are readily determined.

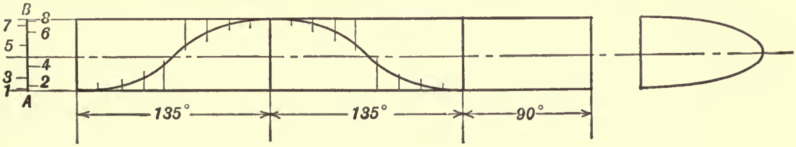


FIG. 284.

It should be noted that the pressure angle increases as the inner end of the follower roller is approached and that the pitch cylinder must pass through the point P, Fig. 284, where the pressure angle is a maximum. To arrive at the outside radius of the cam cylinder, add to the radius of the pitch cam the length of the

contact line, *PQ*, of the cam roller, Fig. 285. The length of the cam chart, Fig. 284 is equal to the outside circumference of the cam cylinder.

When the cam chart, Fig. 284, is wound around the cylinder the line *AB* will be parallel to the axis as shown in Fig. 285. After locating the proper divisions on the end view of the cylinder project lines down and across from corresponding points as at *2*, *2'*

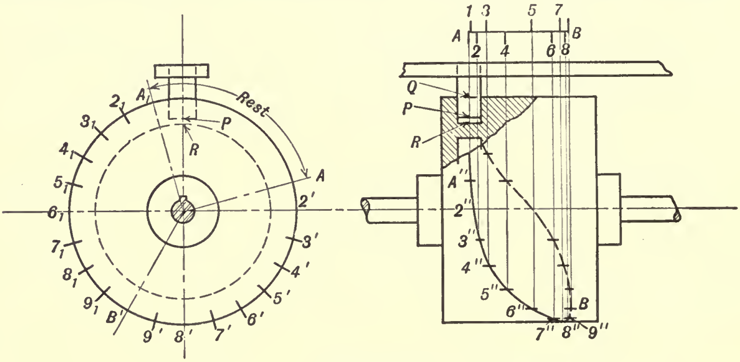


Fig. 285.

and the intersections will be points on the required curve, i.e., the curve that must be followed by the center of the cutting tool.

The following points should be noted:

1. If the follower arm is provided with a roller, clearance should be allowed in the groove, since each side of the groove would tend to make the roller revolve in a different direction.
2. This clearance is taken up at the end of the stroke. A knock results which may be injurious to the machine if running at a high speed.
3. Clearance must be provided between the bottom of the roller and the bottom of the cam groove as shown at *PR*, Fig. 285.
4. If a cylindrical roller is used pure rolling contact does not exist between the working surface of the cam and the follower roller. This condition results in wear.

Pure rolling contact may be secured by using a conical roller as indicated in Fig. 286. This, however, gives a vertical pressure component which tends to force the roller out of the groove and produces friction in the follower guides.

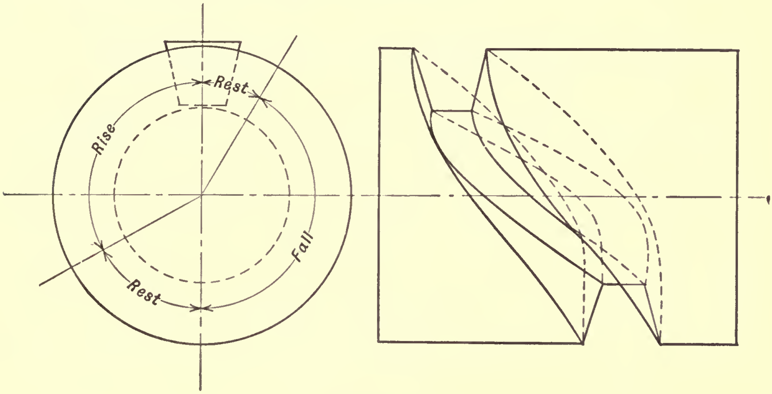


Fig. 286.

5. If a cylindrical roller is used the working surface is a right helicoid while with a conical roller it is an oblique helicoid.

CHAPTER XII

WRAPPING CONNECTORS

232. Introductory.—*Belts, Ropes,* and *Chains* are grouped together as *wrapping connectors.* Such machine elements are used either (*a*) to transmit rotary motion from one link to another or (*b*) to transmit force by direct tension. All three types are used to transmit motion. Belts are not used for the direct transmission of force.

233. Transmission of Motion. Equivalent Link.—Let links *2* and *3*, Fig. 287, represent two pulleys, and let pulley 2 drive

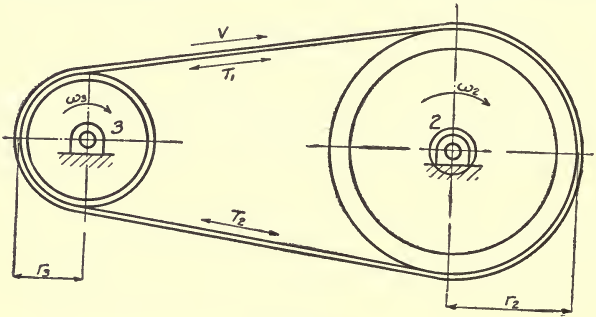


FIG. 287.

pulley 3 by means of a rope, belt, or chain. Then if V is the velocity of the belt

$$V = r_2\omega_2 = r_3\omega_3$$

or

$$\frac{r_2}{r_3} = \frac{\omega_3}{\omega_2},$$

where r_2 and r_3 are the radii to the neutral plane of the belt, provided that there is no slip or stretch of the belt. The neutral plane is usually assumed at half the thickness of the belt, although

it may vary somewhat from this position. Exactly the same motion is obtained by means of the mechanism shown in Fig.

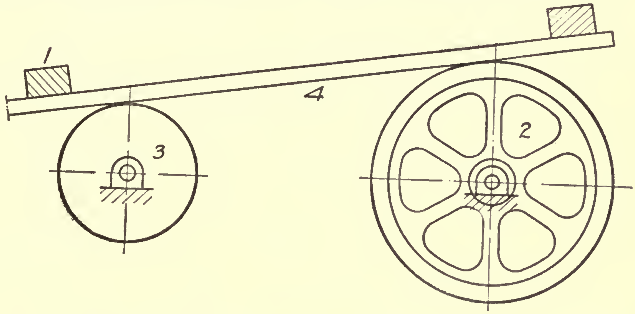


FIG. 288.

288. Here the belt has been replaced by means of a rack, link *4*, meshing with two gear wheels, links *2* and *3*, whose pitch radii are r_2 and r_3. Therefore, for purposes of analysis a belt, rope or chain can be replaced by a rack, provided that there is no slip.

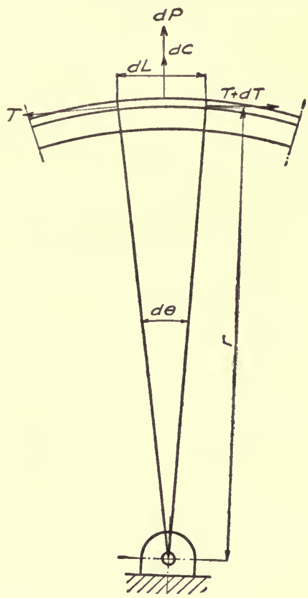


FIG. 289.

Belts and *ropes* depend upon *friction* to furnish the necessary driving force, and there is nearly always some slip. Such drives are therefore unsuitable for use where the velocity ratio transmitted must be accurately maintained. The motion in short is unconstrained. Chains, on the other hand, cannot slip and are therefore much used where exact velocity ratios are required.

234. Friction between Belt and Pulley.—The maximum force to be transmitted by a chain depends only on the strength of the chain. With belts and ropes the maximum force depends not only on the strength of the member, but must not exceed the value which will cause slipping. Consider an element of a belt running on a pulley of radius r, Fig. 289.

Let dL = length of the element;
θ = arc of contact between belt and pulley in radians;
r = radius of pulley;
T_1 = tension on tight side of belt;
T_2 = tension on loose side of belt;
T = tension on element considered;
w = mass of belt per unit length;
μ = coefficient of friction.

Then the mass of the element considered is

$$wdl = wrd\theta. \qquad (1)$$

The forces acting on the element are:

Tension T acting along the tangent to the left;
Tension $T+dt$ acting along the tangent to the right.
Centrifugal force

$$= wrd\theta \cdot r\omega^2 = wr^2\omega^2 d\theta = wV^2 d\theta. \qquad (2)$$

Normal pressure $= dP$ between pulley and belt.
Friction $= \mu dP$ acting along the tangent to the left.

Resolving the tensions into components along the radius and along the tangent,

$$\text{Radial component} = 2T \sin\frac{d\theta}{2} = Td\theta.$$

$$\text{Tangential component} = T \cos\frac{d\theta}{2} = T \text{ to the left,}$$

and

$$(T+dt)\cos\frac{d\theta}{2} = T+dt \text{ to the right.}$$

Then for equilibrium

$$Td\theta = dP + wV^2 d\theta \qquad (3)$$

and

$$dT = \mu dP. \qquad (4)$$

Eliminating dP between Equations (3) and (4)

$$Td\theta = \frac{1}{\mu}dT + wV^2 d\theta. \qquad (6)$$

Or

$$d\theta = \frac{dT}{\mu(T - wV^2)}. \qquad (6)$$

Integrating between limits 0 and θ, T_2 and T_1,

$$\theta=\frac{1}{\mu}\log_e\left(T-wV^2\right)_{T_2}^{T_1}=\frac{1}{\mu}\log_e\frac{T_1-wV^2}{T_2-wV^2}. \quad . \quad . \quad . \quad (7)$$

Or

$$e^{\mu\theta}=\frac{T_1-wV^2}{T_2-wV^2}. \quad . \quad . \quad . \quad . \quad (8)$$

The net driving force acting on the pulley is

$$T_1-T_2=(T_2-wV^2)(e^{\mu\theta}-1). \quad . \quad . \quad . \quad . \quad (9)$$

Equation (9) gives the maximum value of the force which can be transmitted without slip. In addition the tension T_1 must not exceed the safe working strength of the belt. In general the value of T_1-T_2 found by Equation (9) will be different for the two pulleys. In practically all cases the limiting value is to be found at the smaller pulley.

235. Power Transmitted by Belt.—The net pull acting on the circumference of the pulley is T_1-T_2. The work performed per second is

$$W=(T_1-T_2)V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the belt is on the point of slipping

$$T_1-T_2=(T_2-wV^2)(e^{\mu\theta}-1)=(T_1-wV^2)(1-e^{-\mu\theta}) \quad . \quad (2)$$

Then

$$W=(1-e^{-\mu\theta})(T_1V-wV^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Evidently if

$$T_1=wV^2,\ W=0.$$

There is therefore a maximum speed above which power cannot be transmitted. Also if $V=0$, $W=0$. It follows that there must be some speed V between $V=0$ and $V=\sqrt{\frac{T_1}{w}}$ at which the power transmitted becomes a maximum.

Considering T_1 as constant

$$\frac{dW}{dV}=(1-e^{-\mu\theta})(T_1-3wV^2). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

At the speed of maximum power $\frac{dW}{dV}=0$ and therefore

$$T_1=3wV^2$$

or

$$V=\sqrt{\frac{T_1}{3w}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

236. Length of Belts. Cone Pulleys.—In determining the length of a belt two cases arise (*a*) where the belt is crossed, (*b*) where it is uncrossed. In Fig. 290 is shown a crossed belt connecting two pulleys of different diameters. The length of the belt is easily seen to be

$$l=2d\cos\theta+r_1(\pi+2\theta)+r_2(\pi+2\theta)$$

also

$$\sin\theta=\frac{r_1+r_2}{d},$$

therefore

$$l=2d\sqrt{1-\left(\frac{r_1+r_2}{d}\right)^2}+(r_1+r_2)\left((\pi+2\sin^{-1}\frac{r_1+r_2}{d}\right)$$

$$=2\sqrt{d^2-(r_1+r_2)^2}+(r_1+r_2)\left(\pi+2\sin^{-1}\frac{r_1+r_2}{d}\right). \quad . \quad (1)$$

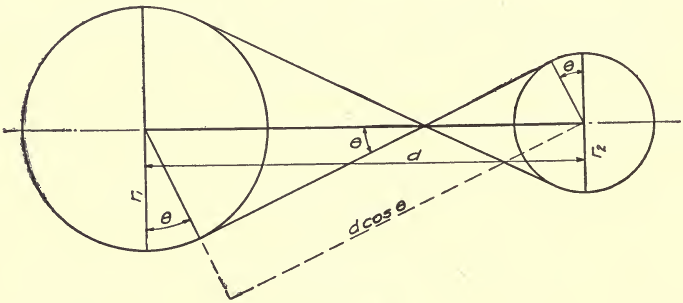


Fig. 290.

It will be noted that the length of the belt depends only on the distance between centers and the *sum* of the radii of the pulleys.

Fig. 291 shows the case of the open belt. Here the length l is given by the equation

$$l=2d\cos\theta+r_1\,(\pi+2\theta)+r_2(\pi-2\theta),$$

where

$$\sin\theta=\frac{r_1-r_2}{d},$$

therefore

$$l=2\sqrt{d^2-(r_1-r_2)^2}+(r_1+r_2)\pi+2(r_1-r_2)\sin^{-1}\frac{r_1-r_2}{d}. \quad (2)$$

In this equation r_1-r_2 appears as well as r_1+r_2.

Equations (1) and (2) are important in the design of cone pulleys. Such pulleys are designed so that the length of the belt should remain constant. Therefore in the case of crossed belts the only condition to be satisfied is that the sum of the diameters of the corresponding pulleys on the two cones should be constant. For open belts the conditions are more complex. For example, suppose that the largest pulley on one cone has a radius of 18 inches and the smallest on the other has a radius of 6 inches, the distance between shaft centers being 8 feet 4 inches. It is required to determine the diameters of two other pulleys whose ratio should be 2 : 1.

$$l=2\sqrt{100^2-(18-6)^2}+(18+6)\pi+2(18-6)\sin^{-1}\frac{18-6}{100}=276.8''.$$

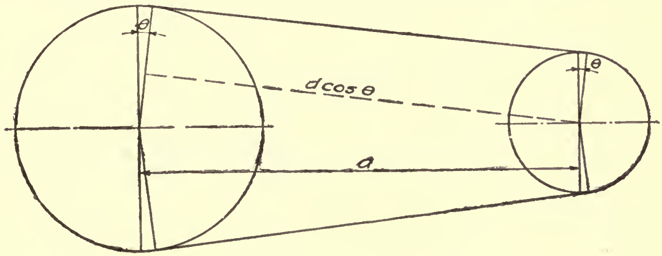


FIG. 291.

Putting $r_1=2r_2$ we have

$$276.8=2\sqrt{100^2-r_2{}^2}+3r_2\pi+2r_2\sin^{-1}\frac{r_2}{100}.$$

This is a transcendental equation and can be solved only by trial. Since, however, θ is a small angle we may write

$$\sin^{-1}\frac{r_2}{100}=\frac{r_2}{100},$$

and also

$$\sqrt{100^2-r_2{}^2}=100-\frac{r_2{}^2}{200}.$$

without introducing any appreciable error.

Then

$$276.8=2\left(100-\frac{r_2{}^2}{200}\right)+3\pi r_2+\frac{2r_2{}^2}{100}=200+3\pi r_2+\frac{r_2{}^2}{100}.$$

Solving $r_2=8.1''$.

237. Arrangement of Twisted Belts. Reversibility.—Belts may be used to transmit motion either between parallel shafts or between non-parallel non-intersecting shafts. The condition for proper running is that the center line of the belt approaching each pulley must lie in the central plane of the pulley. The angle

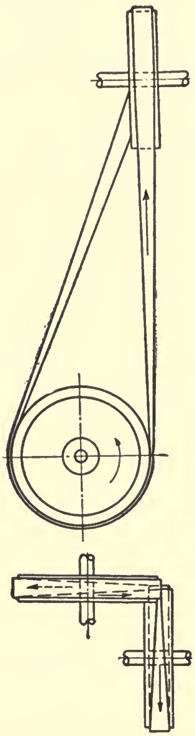


Fig. 292.

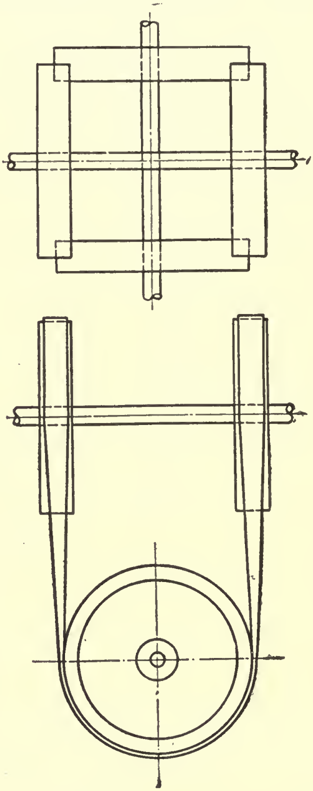


Fig. 293.

at which the belt leaves the pulley is immaterial. For example, the arrangement shown in Fig. 292, will operate satisfactorily as long as the rotation is in the direction shown. If, however, the rotation is reversed, the belt will immediately run off the pulleys.

It is impossible to arrange a reversible belt drive between non-parallel shafts using only two pulleys. By the use of guide

pulleys or idlers reversible drives become feasible. Reversible drives are shown in Figs. 293, 294, and 295. In all these arrangements it will be noted that the center line of the belt entering each pulley lies in the central plane of the pulley. The condition for reversibility is simply that the belt must enter and leave each pulley so that the center line of the belt lies in the central plane of the pulley.

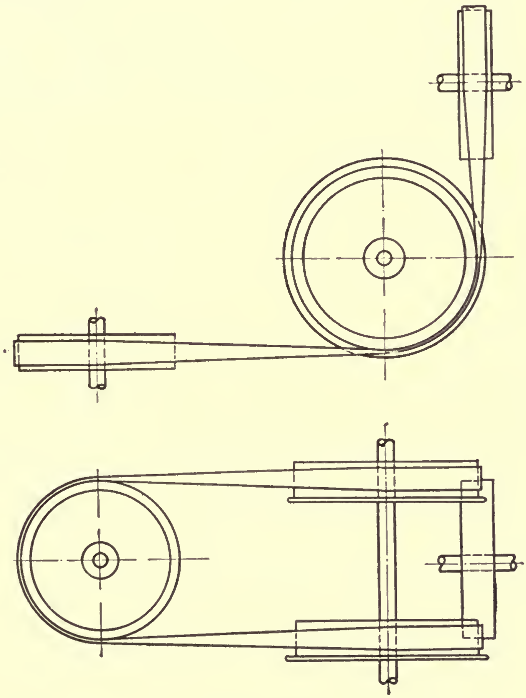
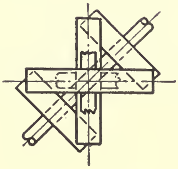


FIG. 294.

238. Idlers.—Idlers are used to guide the belt where the planes of the pulleys do not coincide, to support long belts, and to regulate the tension of the belt. The use of idlers as guide pulleys was illustrated in the preceding article. In the case of long belts idlers are sometimes used to support the belt at intervals and thus prevent an undue amount of sag. In the arrangements shown in Figs. 296, 297, 298, and 299 the idlers serve to regulate



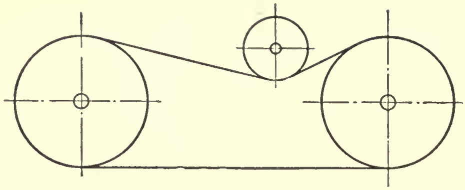


FIG. 296.

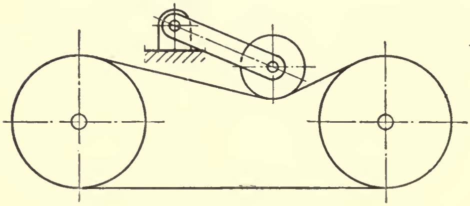


FIG. 297.

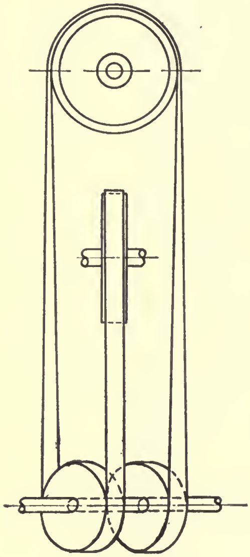


FIG. 295.

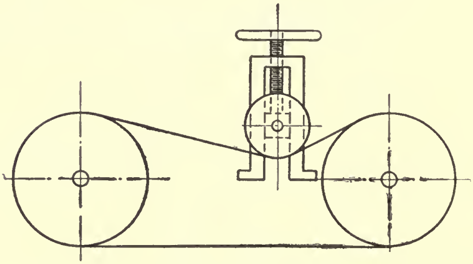


FIG. 298.

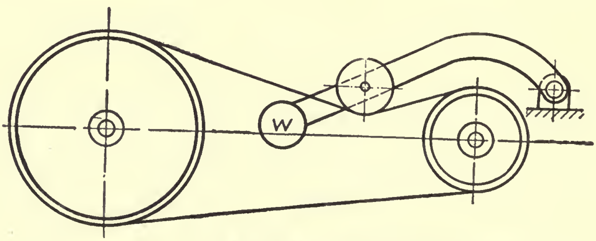


FIG. 299.

the tension, and incidentally to increase the arc of contact between belt and pulley.

239. Crowned Pulleys.—Most belt pulleys are *crowned,* that is, the rims are made convex as shown in Fig. 300. A belt always tends to climb to the highest portion of the pulley surface. Therefore with a crowned pulley the belt tends to center itself instead of running off the pulley. Suppose the belt, Fig. 301, to be running on the left side of the crowned pulley. Owing to the lateral stiffness of the belt the portion which is approaching the pulley is thrown to the right as indicated by the dotted lines. This portion then runs on the pulley in a position nearer the central plane. In other words the belt tends to center itself on the pulley.

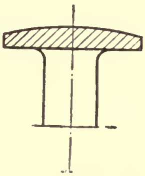


FIG. 300.

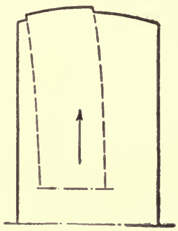


FIG. 301.

240. Materials of Belts.—Most belts are made of leather, but sometimes canvas or other textiles are used. Textile belts are usually treated with some water-proofing material. Rubber belts are sometimes used, the rubber composition being forced into a strong textile material. The rubber is used chiefly to give a bettter grip on the pulley.

Steel belts have been used in Germany with good success. To date such belts have not passed the experimental stage in this country.

241. Rope Drives.—While belts give satisfactory service where the shafts are relatively close together and where the belt is not exposed to the weather, ropes are greatly to be preferred for transmitting power over long distances.

The general principles of rope transmission are similar to those of belt drives, except that since the rope has little lateral stiffness grooved pulleys must be used to keep the rope from running off.

If a number of pulleys are to be driven by ropes there are two systems in general use.

(a) The English or multiple system.
(b) The American or continuous system.

In the multiple arrangement a separate rope is run from a common drum to each driven pulley as indicated in Fig. 302. In the continuous system a single rope is led consecutively over all the pulleys to be driven, as shown in Fig. 303. If only one

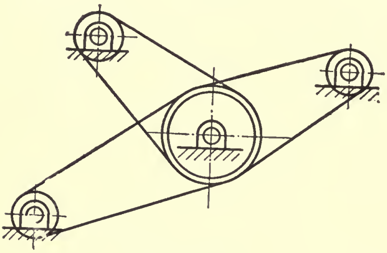


FIG. 302.

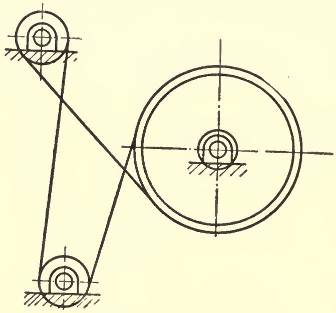


FIG. 303.

driven shaft is used the rope is given a series of loops around the driving and driven drums. In order to carry the rope from one end of the driving drum to the opposite end of the driven drum, a pair of guide pulleys is necessary. The guide pulleys are usually arranged to maintain a constant tension in the slack rope. This device is shown in Fig. 304.[1]

242. Friction between Rope and Pulley.—In order to increase the force which can be transmitted a V-shaped groove is often used, as shown in Fig. 305. The analysis of the forces acting on the rope is similar to that developed for belts. Let dL, Fig. 306 represent the length of an element of the rope. Then the mass of this element is

$$dm = wdL = wrd\theta.$$

The centrifugal force is

$$dC = \frac{dmV^2}{r} = wV^2d\theta.$$

[1] Leutwiler, Machine Design.

If 2ϕ is the angle of the groove, the vertical forces acting on the element are

$$2T\sin\frac{d\theta}{2}-2dP\sin\phi-wV^2d\theta=0,$$

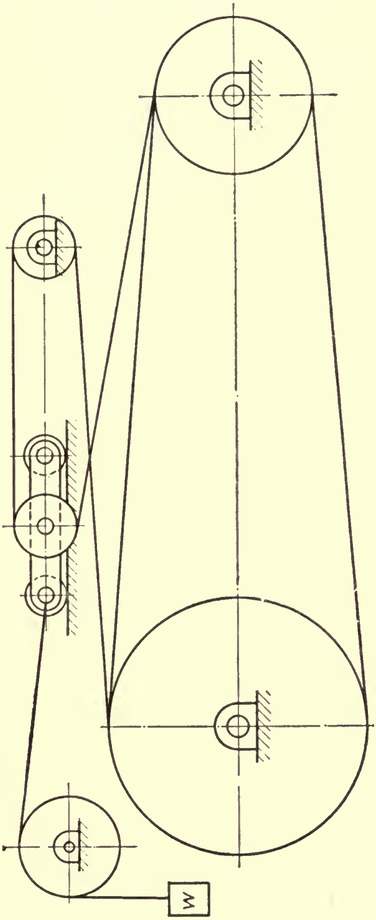


Fig. 304.

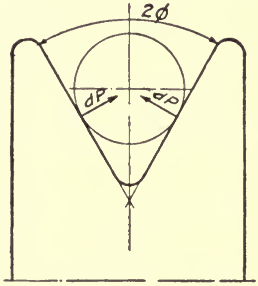


Fig. 305.

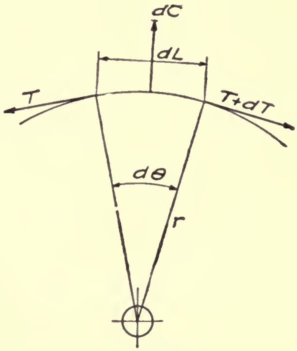


Fig. 306.

and the horizontal forces are

$$dT-2\mu dP=0.$$

Eliminating dP

$$Td\theta-\frac{1}{\mu}\sin\phi dT-wV^2d\theta=0.$$

Therefore

$$\frac{\mu d\theta}{\sin\phi}=\frac{dT}{T-wV^2}.$$

Integrating

$$\mu'\theta=\frac{\mu\theta}{\sin\phi}=\log_e\frac{T_1-wV^2}{T_2-wV^2}.$$

Or

$$e^{\mu'\theta}=\frac{T_1-wV^2}{T_2-wV^2}.$$

It will be noted that this equation is exactly the same as the corresponding equation for belts except that $\frac{\mu}{\sin\phi}$ is substituted for μ.

The angle 2ϕ is usually from 45° to 60°, and therefore $\frac{1}{\sin\phi}$ ranges from 2.61 to 2.0.

243. Chain Drives.—Chain drives are used where the velocity ratio of driving and driven wheels must be accurately maintained, or where the forces involved are too large to be satisfactorily transmitted by belts or ropes.

Chains and chain wheels are made in a large variety of forms according to the service to be performed. It is not the intention of the authors to enter into detailed description of the various makes of chain drives. For such material the reader is referred to the catalogues of the different chain manufacturers.

There is no definite relation between the tensions on the tight and slack sides of a chain drive. The tension on the tight side is limited simply by the strength of the chain, and that on the slack side by the amount of sag which is permissible.

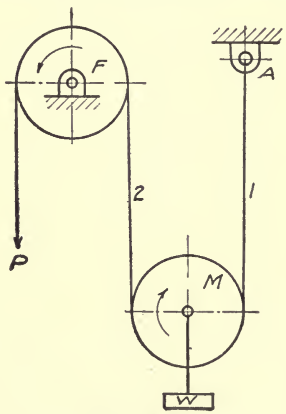


FIG. 307.

244. Hoisting Tackle.—Various combinations of ropes with fixed and movable pulleys are used in hoisting devices. Some of these arrangements are shown in Figs. 307–311.

In Fig. 307 a weight W is hung from a movable pulley M. The end of the rope is fastened at a fixed point A, and the rope passes around

the movable pulley M and over the fixed pulley F. A force P is applied at the free end of the rope to overcome the load W. If the weight W is lifted 1 foot there will evidently be 1 foot of slack in each of the ropes 1 and 2. Therefore 2 feet of rope must be hauled in at P to take up this slack, and the velocity ratio of P to W is 2 to 1.

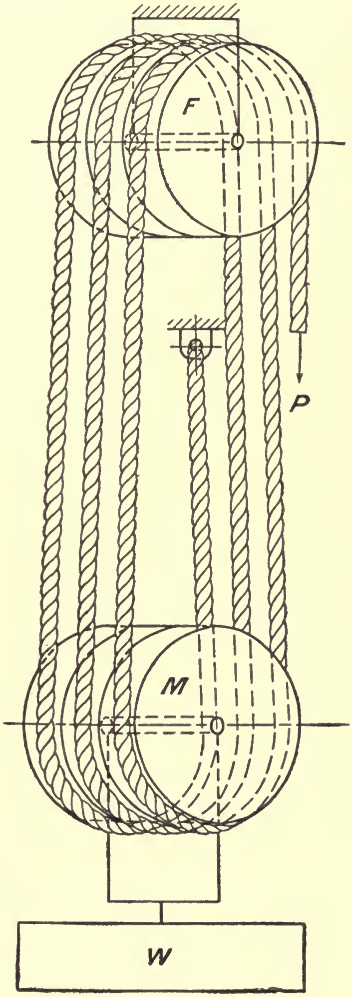


Fig. 308.

In Fig. 308 the movable block M consists of three pulleys. A rope makes three loops around the three pulleys in M and the three pulleys of the fixed block F. If the weight W is lifted 1 foot, each of the six supporting strands is given 1 foot of slack. Therefore, 6 feet of rope must be hauled in at P to take up this slack, and the velocity ratio of P to W is therefore 6 to 1.

In general if the moving pulley is supported by n ropes the velocity ratio is n to 1.

In Fig. 309 the weight W is hung from a movable pulley M_1. A rope passes around M_1, one end being fastened at a fixed point O_2, and the other end to a second moving pulley M_2. The ends of the rope which supports M_2 are similarly fastened at a fixed point O_2 and to a third moving pulley M_3. One end of the rope which supports M_3 is fastened at O_3 and the other end passes over a fixed pulley F. A pull P on the free end of the rope is used to raise the weight W. Evidently to raise the weight 1 foot M_2 must raise 2 feet, M_3 must raise 4 feet, and 8 feet of rope must be hauled in at P.

In Fig. 310 each of the moving pulleys M_1 and M_2 is supported by three ropes. To raise the weight W 1 foot M_2 must rise 3 feet, and 9 feet of rope must be hauled in at P.

The various arrangements shown in Figs. 307–311 can be combined in different ways to give any required velocity ratio.

245. Differential Pulley.—In Fig. 311 the weight W is carried by a movable pulley M. A

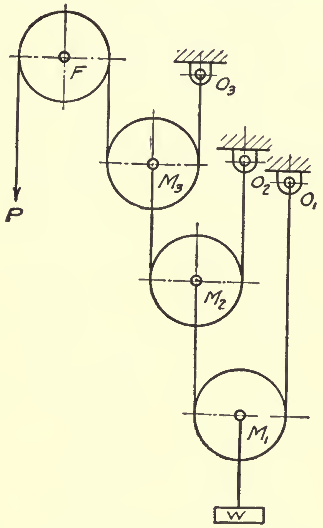


Fig. 309.

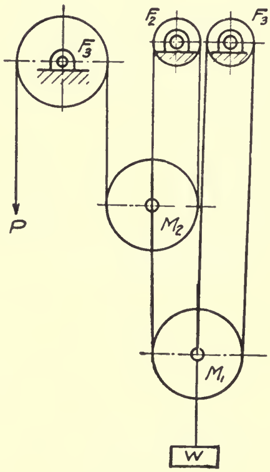


Fig. 310.

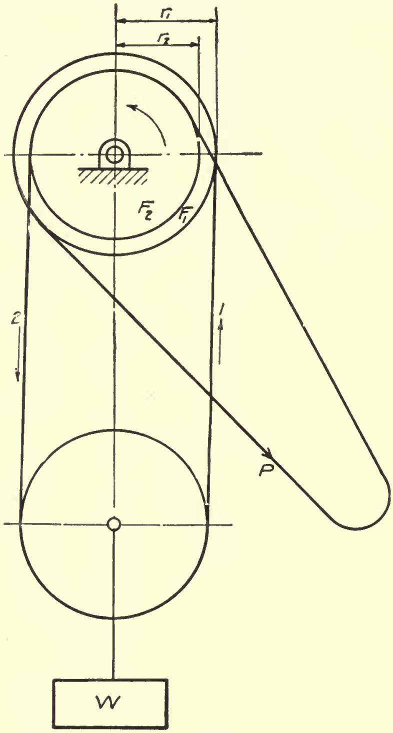


Fig. 311.

chain passes around the movable pulley and the doule fixed pulley F_1F_2 as shown. If a force P be applied on the free

loop of the chain the double fixed pulley will turn in the direction shown.

Let r_1 = radius of F_1,

and

r_2 = radius of F_2.

Then if the block F_1F_2 makes one revolution a length of chain $2\pi r_1$ will be wound on the fixed pulley F_1 and a length $2\pi r_2$ will be wound off the pulley F_2. The loop carrying the movable pulley M will thus be shortened by an amount $2\pi(r_1-r_2)$, and M will be raised a distance $\pi(r_1-r_2)$. The force P acts through a distance $2\pi r_1$. The velocity ratio of P to W is therefore

$$\frac{2\pi r_1}{\pi(r_1-r_2)}=\frac{2r_1}{r_1-r_2}.$$

This device is known as a *differential* hoist, since the ratio of haul to lift depends on the difference r_1-r_2.

NOTE A. IRREGULAR GEARS

In Fig. 312 let the curve A represent the pitch line of an arbitrarily chosen irregular gear rotating about a center P_1. It is required to lay out a second gear B such that B makes one revolution while A makes n revolutions.

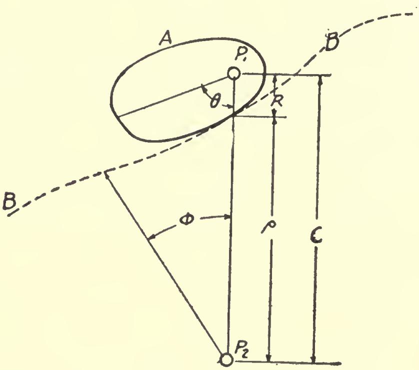


FIG. 312.

The first problem which arises is the determination of the center of rotation of gear B. Let P_2 represent this center. and let the distance

$$P_1P_2=C.$$

Let $R=f\ (\theta)$ be the equation of the pitch line of gear A. Then $\rho=C-R=F\ (\phi)$ will be the equation of the pitch line of gear B.

If gear A rotates through an angle $d\theta$ gear B rotates through an angle $d\phi$ such that

$$d\phi=\frac{R}{C-R}\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since gear B rotates through an angle $\frac{2\pi}{n}$ while gear A makes one revolution,

$$\int_0^{\frac{2\pi}{n}} d\phi=\int_0^{2\pi}\frac{R}{C-R}\,d\theta=\frac{2\pi}{n}. \quad . \quad . \quad . \quad . \quad (2)$$

If the form of the function $R=f(\theta)$ is known Equation (2) can be integrated and theoretically the distance C can be determined. Practically the calculation of C is very difficult even when R is a simple function of θ. As an example take $R=a\sin^2\theta$. Then Equation (2) becomes

$$\frac{2\pi}{n}=\int_0^{\pi}\left[\frac{a}{C}\sin^2\theta+\frac{a^2}{C^2}\sin^4\theta+\frac{a^3}{C^3}\sin^6\theta\ldots\right]d\theta$$

$$=\frac{a}{C}\pi+\frac{3}{4}\frac{a^2}{C^2}\pi+\frac{5}{6}\cdot\frac{3}{4}\frac{a^3}{C^3}\pi+\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\frac{a^4}{C^4}\pi\ldots$$

$$=2\pi\left[\frac{1}{2}\frac{a}{C}+\frac{3.1}{4.2}\frac{a^2}{C^2}+\frac{5.3.1}{6.4.2}\frac{a^3}{C^3}+\frac{7.5.3.1}{8.6.4.2}\frac{a^4}{C^4}\cdot\ .\ .\ .\right]. \quad (3)$$

It can be shown that the solution of Equation (3) is

$$\frac{a}{C}=\frac{2}{n}-\frac{3}{n^2}+\frac{4}{n^3}-\frac{5}{n^4}+\frac{6}{n^5}-\frac{7}{n^6}. \qquad (4)$$

If, for example, $n=4$

$$\frac{a}{C}=\frac{2}{4}-\frac{3}{16}+\frac{4}{64}-\frac{5}{256}+\frac{6}{1024}-\frac{7}{4096}\ldots=0.3611.$$

After the distance C is determined the gears can be laid out according to the method explained in Art. 35, or the equation of the pitch line of gear B can be found by means of Equation (1).

From Equation (1)

$$d\theta=\frac{C-\rho}{\rho}\,d\phi, \qquad (6)$$

or

$$d\phi=\frac{\rho}{C-\rho}\,d\theta. \qquad (7)$$

If θ can be expressed in terms of ρ Equation (7) can be integrated giving the equation of the pitch line of gear B.

Fig. 313 shows the layout of a pair of irregular gears according to the following data:

$$R=a\sin^2\theta$$

where

$$a=1''.$$

Gear A makes four revolutions to one revolution of B.

It is evident from the forms of the gears that it would be impracticable to supply them with teeth. Such pitch lines, however, could be made to roll together without difficulty.

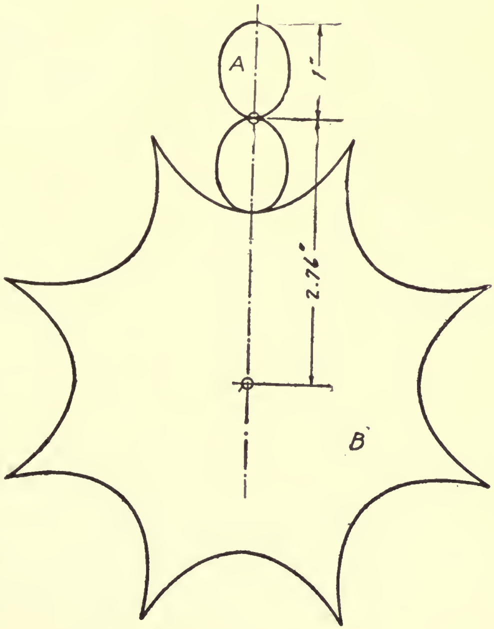


Fig. 313.

NOTE B. PROPOSITIONS ON VELOCITY POLYGONS

Proposition 1.—Given the velocity image of a link AB, Fig. 314, to find the instantaneous center of rotation of that link. Let ab be the revolved image of AB and let O be the pole of velocities. From A and B draw lines parallel to aO and bO respectively. The intersection P of these lines is the instantaneous center. The triangle ABP is similar to Oab. O is the image of a point on the link which has no velocity—that is the instantaneous center.

If the image of AB reduces to a point (ab) the instantaneous center is at infinity in the direction $O(ab)$.

Proposition 2.—Given the images of two links, AB and CD, Fig. 315, to find the instantaneous center of relative motion.

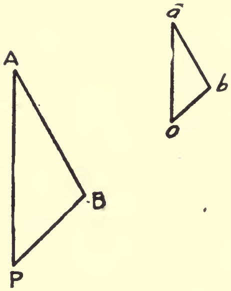


Fig. 314.

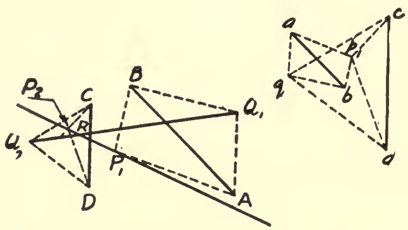


Fig. 315.

Let ab and cd be the revolved images of AB and CD. Choose at random any point p and draw pa, pb, pc, and pd. From A and B draw lines parallel to pa and pb respectively. Let these lines intersect at P_1. From C and D, draw lines parallel to pc and pd intersecting at P_2. Join P_1P_2. Choose a second random point q and repeat the process. The intersection R of P_1P_2 and Q_1Q_2 is the instantaneous center of relative motion.

Proof.—p and q may be regarded as the images of P_1P_2, Q_1Q_2. The relative motion is unaffected by any motion given to both links simultaneously. Let O be the pole of velocities. Suppose the whole system to be given a velocity pO, thus bringing P_1 and P_2 to rest. P_1 and P_2 thus become the instantaneous centers for the two links, and the center of relative velocities must lie

on the line P_1P_2. Similarly the center of relative velocity must lie on the line Q_1Q_2.

Proposition 3.—Given the velocity polygon for a chain to construct the velocity polygon when a different link is held stationary. The method of doing this is best shown by an example. Given the six-link chain shown in Fig. 316. Let the velocity of A be known. The velocity polygon is readily constructed as shown. The problem is to construct directly a second polygon for the same chain with link *3* held stationary. First, locate P the instantaneous center 13. Link *3* can be brought to rest

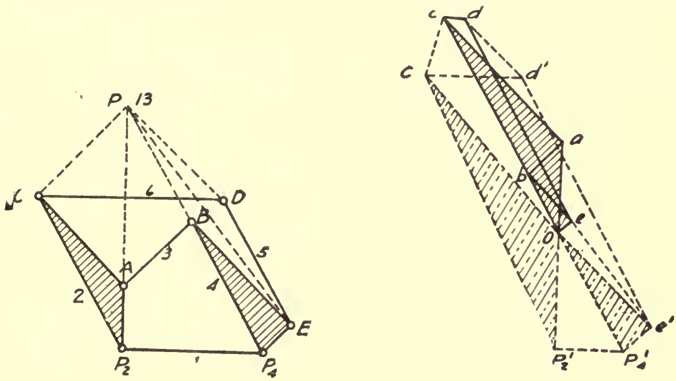


FIG. 316.

by giving the whole mechanism a rotation $= -\omega_{31}$ about P. Then each point will be given an additional velocity which is equal to ω_{31} times the distance of the point from the center P. The direction of this velocity (revolved) is parallel to the line from P to the point in question. Thus, the additional velocity of C is $PC\omega_{31}$.

$$\omega_{31} = \frac{Oa}{PA}.$$

The result of adding this rotation is to bring the images of A and B to O, the image of C to c', of D to d', E to e', P_2 to p_2', P_4 to p_4'. The polygon with link *3* held stationary is then as shown in dotted lines.

NOTE C. LOCUS OF THE CENTER OF ACCELERATION

Consider the epicyclic gear train shown in Fig. 317. Let the arm OA, link *2*, revolve about O with angular velocity Ω and angular acceleration Ω'. This will cause the gear *3* to revolve with angular velocity $\omega=\frac{L}{r}\Omega$ and angular acceleration $\omega'=\frac{L}{r}\Omega'$. Consider any point P on gear *3*, at a distance ρ from the center. Let the angle $OAP=\theta$

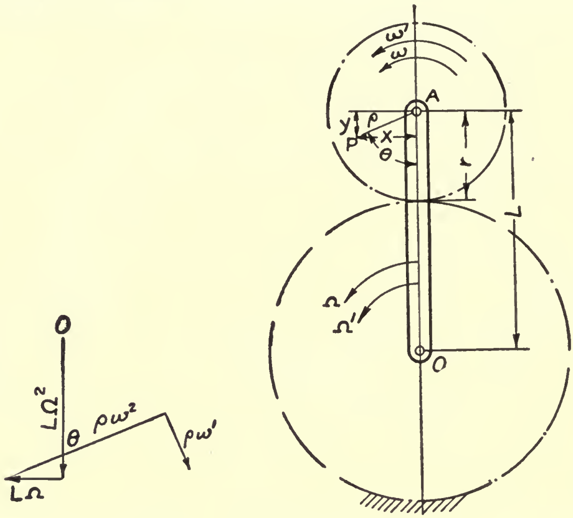


Fig. 317.

Then the acceleration of P is given by the equation:

$$A_p=A_a+A_{pa}=L\Omega^2+L\Omega'+\rho\omega^2+\rho\omega'.$$

For the center of acceleration of the gear *3*:

$$A_p=0.$$

Resolve the four components of A_p in horizontal and vertical directions. Then for the horizontal components:

$$L\Omega'-\rho\omega^2\sin\theta-\rho\omega'\cos\theta=0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For the vertical components:

$$L\Omega^2-\rho\omega^2\cos\theta+\rho\omega'\sin\theta=0. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let $\qquad \rho\sin\theta=x \quad$ and $\quad \rho\cos\theta=y.$

Then since $L\Omega' = r\omega'$ and $L\Omega^2 = \frac{r^2}{L}\omega^2.$

$$r\omega' - x\omega^2 - y\omega' = 0, \quad \ldots \ldots \quad (3)$$

$$\frac{r^2\omega^2}{L} - y\omega^2 + x\omega' = 0. \quad \ldots \ldots \quad (4)$$

Let
$$\frac{\omega^2}{\omega'} = k.$$

Then from Equation (3)
$$r - y - xk = 0.$$

Therefore
$$k = \frac{r-y}{x}. \quad \ldots \ldots \quad (5)$$

From Equation (4)
$$\frac{r^2}{L}k - yk + x = 0. \quad \ldots \ldots \quad (6)$$

Hence
$$\frac{r^2}{L}\left(\frac{r-y}{x}\right) - y\left(\frac{r-y}{x}\right) + x = 0,$$

or
$$\frac{r^3}{L} - \frac{r^2}{L}y - ry + y^2 + x^2 = 0. \quad \ldots \ldots \quad (7)$$

Equation (7) is the equation of a circle, which is the locus of P. The center of this circle is on the vertical (y) axis. The equation can be put in the form:

$$x^2 + (y-b)^2 = a^2$$

where a is the radius of the circle;

$$(y-b)^2 = y^2 - 2by + b^2.$$

Therefore

$$-2b = -\frac{r^2}{L} - r,$$

$$b = \frac{r}{2}\left(\frac{r+L}{L}\right) \quad \text{and} \quad b^2 = \frac{r^2}{4}\left(\frac{r+L}{L}\right)^2,$$

$$a^2 = \frac{r^2}{4}\left(\frac{r+L}{L}\right)^2 - \frac{r^3}{L} = \left(\frac{r}{2}\right)^2\left[\left(\frac{r+L}{L}\right)^2 - \frac{4r}{L}\right]$$

$$= \left(\frac{r}{2}\right)^2\left[\frac{r^2+2Lr+L^2}{L^2} - \frac{4rL}{L^2}\right]$$

$$= \left(\frac{r}{2}\right)^2\left(\frac{L-r}{L}\right)^2$$

$$a = \frac{r}{2}\cdot\frac{L-r}{L} = \frac{r}{2}\cdot\frac{R}{r+R}. \quad \ldots \ldots \quad (8)$$

The position of P on the circle depends on the value of k. If $k=0$, then, from (5)

$$r-y=0 \quad \text{or} \quad y=r.$$

That is, the circle must be tangent to the fixed gear at B.

It has been shown in Art. 32 that the motion of any rigid link in a machine may be reproduced with absolute accuracy by rolling the moving centrode on the fixed centrode. Let FF and MM, Fig. 318, be the fixed and moving centrodes for the given link. Let O be the center of curvature of the fixed centrode and A that of the moving centrode. Then for a very small displacement a rigid link might be put in connecting O and A without in the slightest degree altering the motion. If this is done it is evident that the argument developed for the epicyclic gear train applies equally well to the rolling centrodes. The radii r and R now become the radii of curvature of the centrodes. The locus of the center of acceleration for the moving link is then a circle $BCDE$ whose radius is: $\frac{r}{2}\frac{R}{r+R}$ and which is tangent to the two centrodes at B.

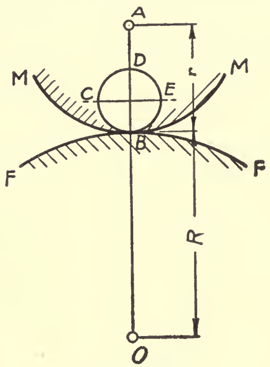


Fig. 318.

NOTE D. ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

Definitions.—An ordinary differential equation of the nth order is an equation expressing a relation between an independent variable x, a dependent variable y, and the first n derivatives of y with respect to x. The general form of such an equation is

$$F\left(x,\, y,\, \frac{dy}{dx},\, \frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n}\right)=0.$$

A linear differential equation is a differential equation in which only the first powers of the dependent variable and its various derivatives occur, and in which no product of two derivatives or of the dependent variable with any of the derivatives appears.

The general form of a linear differential equation of the nth *order* is

$$\frac{d^ny}{dx^n}+P_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+P_{n-2}\frac{d^{n-2}y}{dx^{n-2}}\ldots+P_2\frac{d^2y}{dx^2}+P_1\frac{dy}{dx}+P_0y=R,$$

where P_{n-1}, $P_{n-2}\ldots P_0,R$ are functions of x only.

A linear differential equation with constant coefficients is one in which the functions P_{n-1}, $P_{n-2}\ldots P_0$ are replaced by constants. Equations of this type appear repeatedly in engineering problems such as those treated in the text. The object of this note is to enable students to handle such equations intelligently.

Solution of Differential Equations.—A differential equation is said to be solved when the dependent variable is expressed as a function of the independent variable. Thus

$$f(x, y)=0, \quad \ldots\ldots\ldots \quad (1)$$

is a solution of the differential equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\ldots\frac{d^ny}{dx^n}\right)=0, \quad \ldots\ldots \quad (2)$$

provided that when the various derivatives are formed and substituted in Equation (2) the latter equation is satisfied.

Solutions are classified as

(1) General solutions.
(2) Particular solutions.
(3) Complete solutions.
(4) Singular solutions.

Complete solutions and *singular* solutions are not required in any of the problems arising in the text, and therefore will not be considered. The *general* solution of a differential equation always includes a *number of arbitrary constants equal to the order of the highest derivative occurring in the equation.* A *particular solution* is derived from the general solution by giving particular values to the arbitrary constants.

For example:

$$\frac{d^4y}{dx^4}=12x^2 \quad \ldots\ldots\ldots \quad (3)$$

is a differential equation of the fourth order. By successive integrations we obtain

$$\frac{d^3y}{dx^3}=4x^3+A,$$

$$\frac{d^2y}{dx^2}=x^4+Ax+B,$$

$$\frac{dy}{dx}=\frac{1}{5}x^5+\frac{1}{2}Ax^2+Bx+C,$$

$$y=\frac{1}{30}x^6+\frac{1}{6}Ax^3+\frac{1}{2}Bx^2+Cx+D, \quad . \quad . \quad . \quad . \quad (4)$$

where A, B, C and D are arbitrary constants. Equation (4) is the general solution of Equation (3). If the values $A=12$, $B=6$, $C=0$ and $D=9$, are substituted in Equation (4) we obtain a particular solution,

$$y=\tfrac{1}{30}x^6+2x^3+3x^2+9.$$

Linear Differential Equation, Second Member Zero.—Let

$$\frac{d^ny}{dx^n}+A\frac{d^{n-1}y}{dx^{n-1}}+B\frac{d^{n-2}y}{dx^{n-2}}+\ldots L\frac{d^2y}{dx^2}+M\frac{dy}{dx}+Ny=0 \quad . \quad (1)$$

be chosen as a differential equation which is to be solved. Assume that a solution can be written in the form

$$y=C_1e^{m_1x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Forming the various derivatives

$$\frac{dy}{dx}=C_1m_1e^{m_1x},$$

$$\frac{d^2y}{dt^2}=C_1m_1^2e^{m_1x},$$

. . . .

. . . .

$$\frac{d^ny}{dx^n}=C_1m_1^ne^{m_1x}.$$

Substituting these values in Equation (1),

$$C_1e^{m_1x}(m_1^n+Am_1^{n-1}+Bm_1^{n-2}+\ldots Lm_1^2+Mm_1+N)=0. \quad . \quad (3)$$

Equation (3) is satisfied provided that m_1 is a root of the equation

$$m^n+Am^{n-1}+Bm^{n-2}+\ \ldots\ Lm^2+Mm+N=0. \quad . \quad . \quad (4)$$

Equation (4) is called the *auxiliary* equation. Equation (2) is a solution of Equation (1). It is not, however, a *general* solution, since it contains only one arbitrary constant C_1. But Equation (4) has n roots m_1, $m_2 \ldots m_n$. Any of these values satisfies Equation (4) and therefore also satisfies Equation (3). The general solution of Equation (1) is then

$$y=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}\ \ldots\ C_ne^{m_nx}. \quad . \quad . \quad (5)$$

Character of the Roots.—The roots m_1, $m_2 \ldots m_n$ may be either real or imaginary. If all the roots are real and different, Equation (5) is the simplest form of the general solution. If a pair of the roots, say m_1 and m_2, are conjugate imaginaries, the general solution can be thrown into a more convenient form. Let

$$m_1=a+bi,$$

and

$$m_2=a-bi,$$

where $i=\sqrt{-1}$, then

$$C_1e^{m_1x}+C_2e^{m_2x}=e^{ax}(C_1e^{bix}+C_2e^{-bix}).$$

Now

$$\cos bx=\tfrac{1}{2}(e^{bix}+e^{-bix})$$

and

$$\sin bx=\frac{i}{2}(e^{bix}-e^{-bix}),$$

or

$$i \sin bx-\tfrac{1}{2}(e^{bix}-e^{-bix}).$$

Therefore

$$\cos bx+i\sin bx=e^{-b}\quad ,$$

and

$$\cos bx-i\sin bx=e^{bix}.$$

Hence

$$C_1e^{m_1x}+C_2e^{m_2x}=e^{ax}[(C_1+C_2)\cos bx+(C_2-C_1)\,i\sin bx]$$
$$=e^{ax}[C'_1\cos bx+C'_2\sin bx],$$

where

$$C'_1=C_1+C_2$$

and

$$C'_2=(C_2-C_1)i.$$

Since C_1 and C_2 are arbitrary constants and may be real or imaginary, C'_1 and C'_2 are also arbitrary constants. Therefore if two of the roots m_1 and m_2 form a pair of conjugate imaginaries,

$$m_1=a+bi,$$

$$m_2=a-bi,$$

the corresponding terms of the general solution become

$$C_1e^{ax}\cos bx,$$

and

$$C_2e^{ax}\sin bx.$$

The general solution now becomes

$$y=e^{ax}(C_1\cos bx+C_2\sin bx)+C_3e^{m_3x}\ \ldots\ C_ne^{m_nx}.$$

If two or more of the roots m_1, m_2 ... m_n are equal, the form of the general solution must be somewhat altered. For example, if the roots m_1, m_2 and m_3 are equal the solution

$$\begin{aligned} y&=C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+\ \ldots\ C_ne^{m_nx} \\ &=(C_1+C_2+C_3)\,e^{m_1x}+C_4e^{m_4x}+\ \ldots\ C_ne^{m_nx}, \end{aligned}$$

is no longer a general solution, since the sum $C_1+C_2+C_3$ is simply *one* arbitrary constant, and the number of arbitrary constants is thus reduced to $n-2$. In this case the general solution takes the form

$$y=(C_1+C_2x+C_3x^2)\,e^{m_1x}+C_4e^{m_4x}+\ \ldots\ C_ne^{m_nx}.$$

As this form does not occur in any of the problems considered in the text the proof will not be developed here.

Second Member not Zero. Complementary Function. Particular Integral.—The most general form of linear differential equation with constant coefficients is

$$\frac{d^ny}{dx^n}+A\frac{d^{n-1}y}{dx^{n-1}}+B\frac{d^{n-2}y}{dx^{n-2}}+\ \ldots\ L\frac{d^2y}{dx^2}+M\frac{dy}{dx}+Ny=R.\quad .\quad (1)$$

where A, B ... N are constants and R is a function of x. To solve equation (1) put

$$y=y_0+P.\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad (2)$$

Then

$$\frac{d^ny_0}{dx^n}+A\frac{d^{n-1}y_0}{dx^{n-1}}+B\frac{d^{n-2}y_0}{dx^{n-2}}+\ldots L\frac{d^2y_0}{dx^2}+M\frac{dy_0}{dx}+Ny_0+$$

$$\frac{d^nP}{dx^n}+A\frac{d^{n-1}P}{dx^{n-1}}+B\frac{d^{n-2}P}{dx^{n-2}}+\ldots L\frac{d^2P}{dx^2}+M\frac{dP}{dx}+NP=R\ . \quad (3)$$

Let

$$\frac{d^ny_0}{dx^n}+A\frac{d^{n-1}y_0}{dx^{n-1}}+B\frac{d^{n-2}y_0}{dx^{n-2}}+\ldots L\frac{d^2y_0}{dx^2}+M\frac{dy_0}{dx}+Ny_0=0, \quad (4)$$

and

$$\frac{d^nP}{dx^n}+A\frac{d^{n-1}P}{dx^{n-1}}+B\frac{d^{n-2}P}{dx^{n-2}}+\ldots L\frac{d^2P}{dx^2}+M\frac{dP}{dx}+NP=R\ . \quad (5)$$

Equation (4) is called the *complementary equation* and y_0 is called the *complementary function*. As shown previously

$$y_0=C_1e^{m_1x}+C_2e^{m_2x}+\ldots C_ne^{m_nx}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where m_1, m_2 . . . m_n are the roots of the *auxiliary* equation

$$m^n+Am^{n-1}+Bm^{n-2}+\ldots Lm^2+Mn+N=0.$$

It will be noted that y_0 contains n arbitrary constants. The general solution is therefore

$$y=y_0+P,$$

where P is any *particular* solution of Equation (1) or Equation (5), that is a solution containing no arbitrary constants.

The form of the particular solution P depends on the form of the function R. Methods for finding P are developed in standard works on differential equations. In the text the only cases which arise are those where R is a constant or a polynomial in powers of x. For such cases the method of finding the particular integral is best illustrated by an example.

Let

$$\frac{d^4y}{dx^4}-A^4y=Bx^2+Cx+D \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equation to be solved.

The complementary equation is

$$\frac{d^4y_0}{dx^4}-A^4y_0=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the auxiliary equation is

$$m^4-A^4=0. \qquad (3)$$

The four roots of Equation (3) are

$$m_1=A,$$

$$m_2=-A,$$

$$m_3=A\sqrt{-1},$$

$$m_4=-A\sqrt{-1}.$$

The complementary function is therefore

$$y_0=C_1e^{Ax}+C_2e^{-Ax}+C_3\cos Ax+C_4\sin Ax. \qquad (4)$$

To find the particular integral P write

$$P=a+bx+cx^2+dx^3+ex^4+fx^5+gx^6. \qquad (5)$$

Then

$$\frac{d^4P}{dx^4}=24\,e+120fx+360gx^2 \qquad (6)$$

and

$$\frac{d^4P}{dx^4}-A^4P=24e+120fx+360gx^2\ldots$$

$$-A^4a-A^4bx-A^4cx^2\ldots=Bx^2+Cx+D. \qquad (7)$$

Equating coefficients

$$24e-A^4a=D.$$

$$120f-A^4b=C.$$

$$360g-A^4c=B.$$

$$840h-A^4d=0, \text{ etc.}$$

These equations admit of an infinite number of solutions. The simplest solution, however, is

$$-A^4a=D,$$

$$-A^4b=C,$$

$$-A^4c=B,$$

and

$$d=e=f=g\ldots=0.$$

Therefore

$$a=-\frac{D}{A^4},$$

$$b=-\frac{C}{A^4},$$

$$c=-\frac{B}{A^4},$$

and

$$P=-\frac{1}{A^4}(Bx^2+Cx+D). \quad . \quad . \quad . \qquad (8)$$

Finally

$$y=y_0+P=C_1e^{Ax}+C_2e^{-Ax}+C_3\cos Ax+C_4\sin Ax$$

$$-\frac{1}{A^4}(Bx^2+Cx+D). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Conclusion.—The solution of an ordinary linear differential equation with constant coefficients consists of the following steps:

(1) Write the *complementary equation* by making the second member equal to zero.

(2) Write the *auxiliary equation* and find the n roots m_1, m_2, m_3 . . . m_n.

(3) Write the complementary function

$$y_0=C_1e^{m_1x}+C_2e^{m_2x}+\ .\ .\ .\ C_ne^{m_nx}.$$

(4) Find a particular integral P.

(5) Add P and y_0. This sum gives the general solution of the equation.

For further discussion of the properties and solution of differential equations, the reader is referred to the standard works on the subject.

NOTE E.* INVESTIGATION OF FORCES IN GASOLINE ENGINE

Figs. 319 to 332 and Tables I to IX give the results of an investigation of the forces in a standard six-cylinder motor, as determined by the students of the junior class at the University of Illinois.

Data.—The data for the engine as determined by actual measurements are as follows:

Diameter of cylinder, $3\frac{1}{8}$ inches.
Stroke, $4\frac{1}{4}$ inches.
Clearance, 26.1 per cent.
Weight of piston, 1.94 pounds.
Weight of rod, 1.75 pounds.
Length of rod, $8\frac{1}{8}$ inches.
Distance from wrist pin to center of gravity of rod, $6\frac{5}{16}$ inches.
Radius of gyration of rod 2.95 inches.

The following data were taken from information furnished by the manufacturers:

R.P.M. = 1500.
Intake opens 15° after beginning of suction troke.
Intake closes 42° after beginning of compression stroke.
Exhaust opens 45° before end of working stroke.
Exhaust closes 10° after beginning of suction stroke.
Ignition takes place when crank is on dead center.
Pressure when exhaust closes atmospheric.
Explosion pressure 325 pounds per square inch.

The following data were assumed:

Equation of compression curve $PV^{1.28} = \text{constant}$.
Equation of expansion curve $PV^{1.23} = \text{constant}$.
Theoretical explosion pressure 392 pounds per square inch.

Timing Diagram. Indicator Card. Total Pressure Card.—With the given data the timing diagram, Fig. 319, was first drawn, showing the succession of events in the cylinder. Next the positions of the piston corresponding to the different events were

* The original drawings from which the following illustrations were made were laid out to a scale of 500 pounds = 1 inch. If the reader desires to determine the scale of any figure, a comparison of any numbered ordinate of that figure with its accompanying table will be necessary.

determined and laid down in Fig. 320. On the stroke of the piston as a base, the indicator diagram, Fig. 320, is drawn. The calculations for the pressure on the compression and working strokes are given in Tables I and II. The pressures are next multiplied by the area of the piston, and the resulting forces are plotted on a scale 500 pounds = 1 inch, as the total pressure diagram Fig. 321. The last two columns of Tables I and II give the total pressure, and the ordinate of the total pressure diagram, Fig. 321.

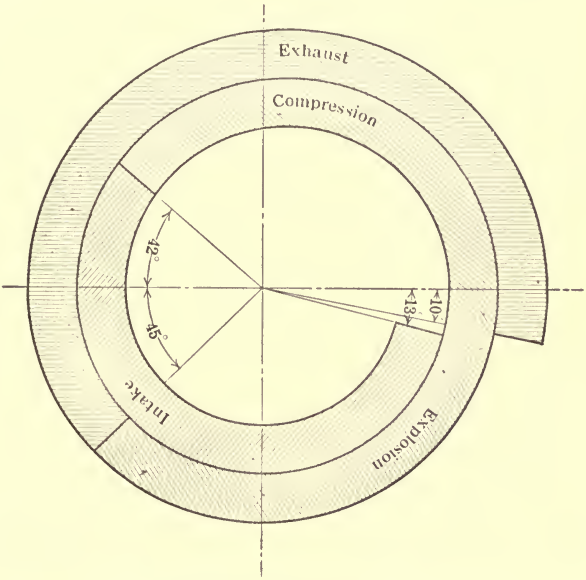


Fig. 319.—Timing Diagram.

Inertia Forces. Net Piston Forces.—The acceleration of the piston is determined by Klein's construction for each 15° of revolution of the crank, and the results are checked by calculation from the equation

$$A = r\omega^2\left(\cos\theta + \frac{r}{L}\cos 2\theta\right).$$

The calculations are given in Table III. Multiplying the accelerations by the mass of the piston as shown in Table IV, the inertia force of the piston is now determined, and is plotted to scale

1 inch = 500 pounds, in Fig. 322. In the same figure the inertia force of the rod is determined by the method of Art. 93 and is plotted in its proper position to a scale 3 inches = 500 pounds.

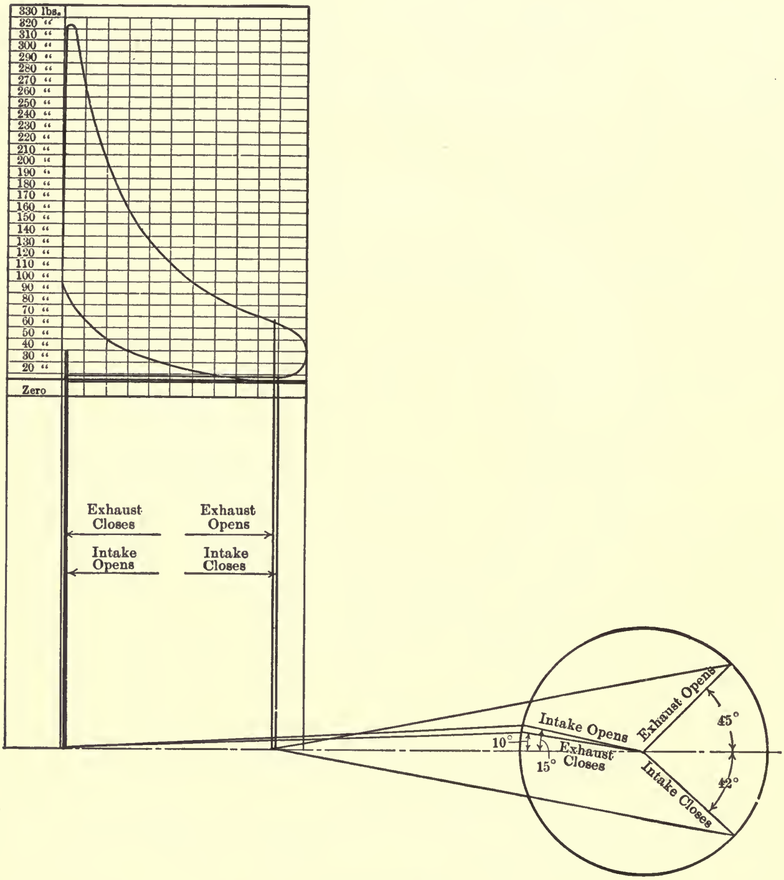


Fig. 320.

A smooth curve drawn through the ends of the vectors representing the rod inertia force shows the variation of this force. Table V gives the calculations for the rod inertia force. The paths of the center of gravity G and of the auxiliary point m_2 of the kinetically equivalent system are also shown in Fig. 322.

In Fig. 323 the total pressure diagram, Fig. 321, is developed on a base line representing the total motion of the piston (disregarding reversals). In this figure forces tending to help the motion of the piston are plotted above the base line, and forces resisting the motion are plotted below the base line.

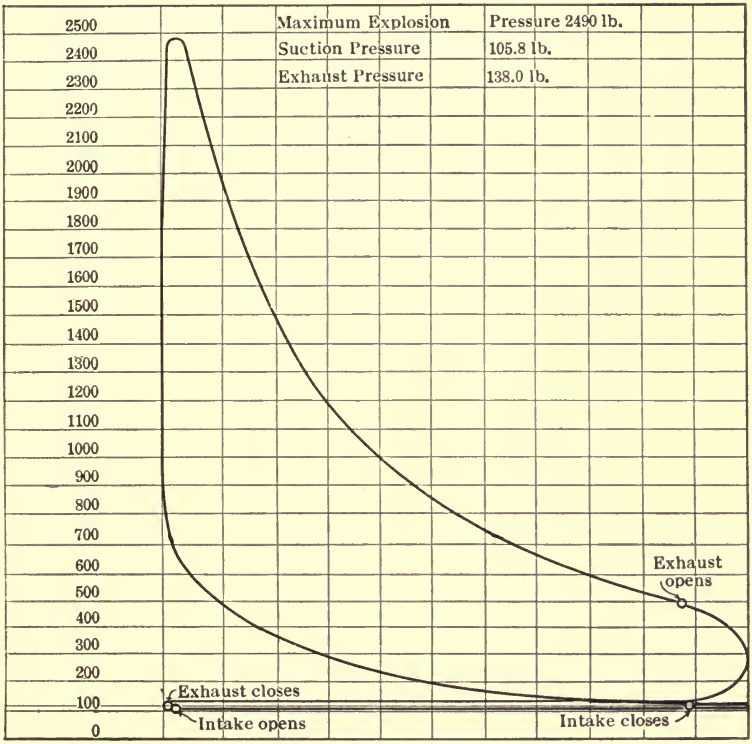


FIG. 321.—Total Pressure Card

The force diagram, Fig. 323, is next transferred to Fig. 324, and there combined with the curve representing the inertia force of the piston. The resultant net piston force is shown by the heavy lines.

Wrist Pin Pressure. Side Thrust.—The total pressure on the wrist pin is the combination of the thrust in the connecting rod with that portion of the inertia force of the rod which acts at the wrist pin. The resultant pressure is found by the method

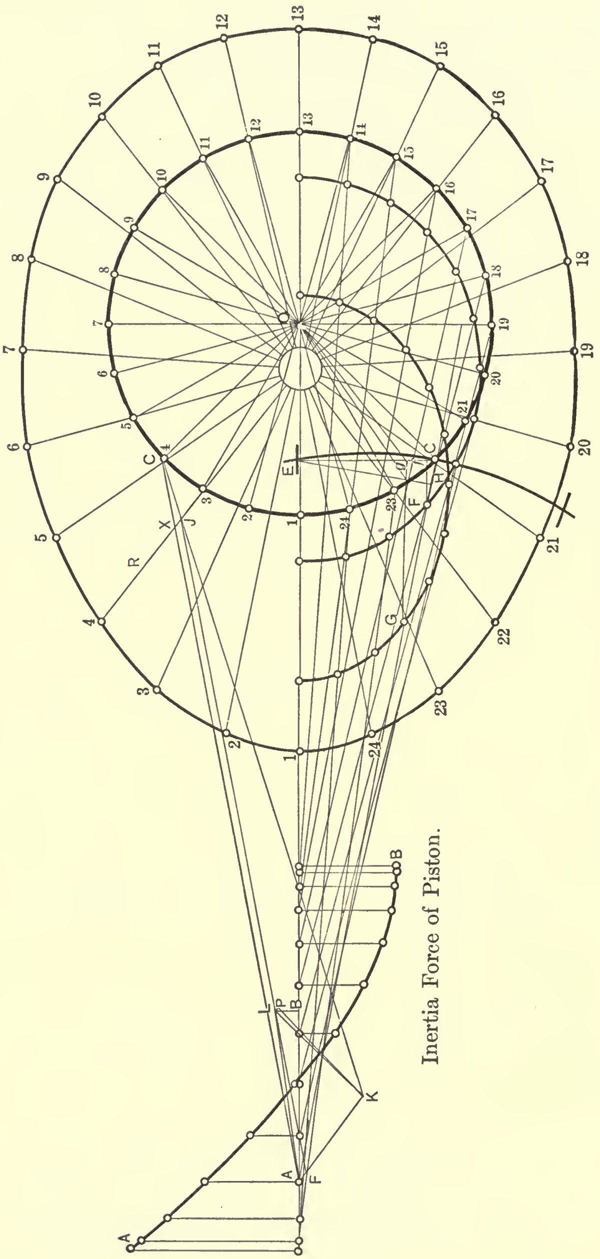


FIG. 322.—Inertia Force of Connecting Rod.

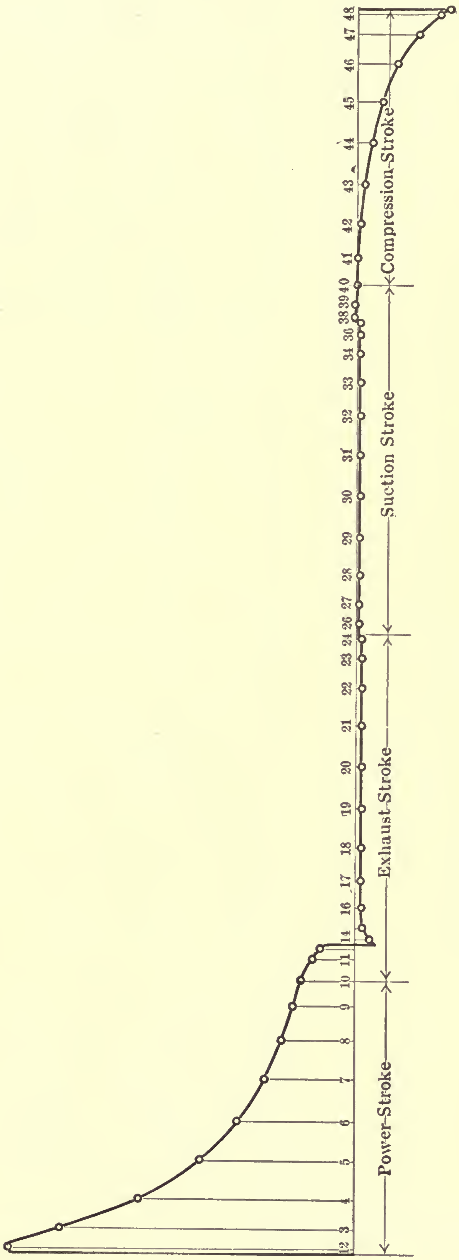


FIG. 323.—Development of Card along a Horizontal Line.

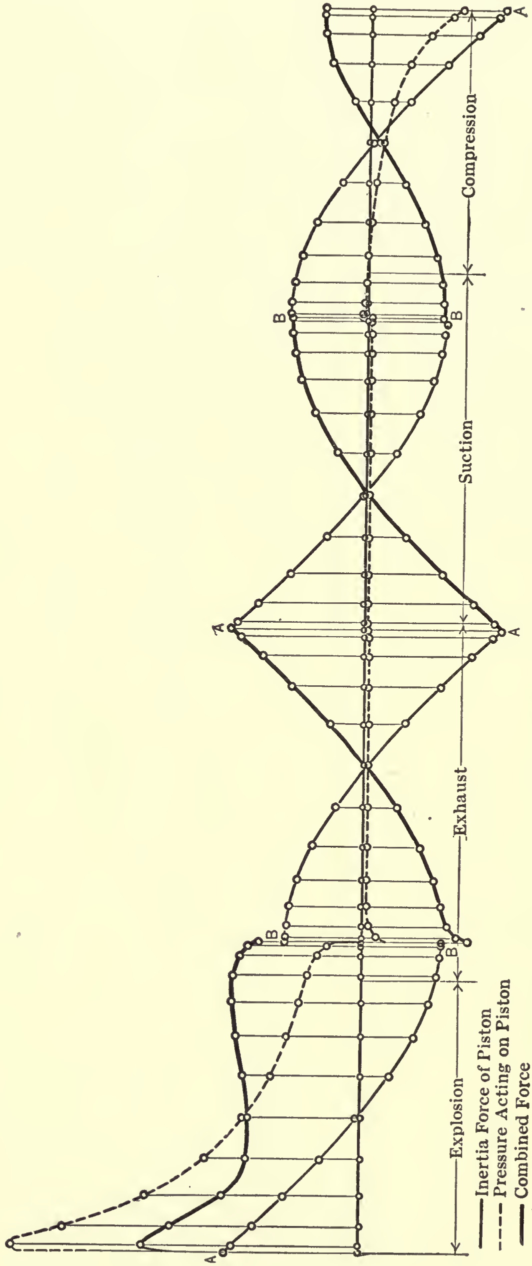


FIG. 324.—Diagram of Combined Forces.

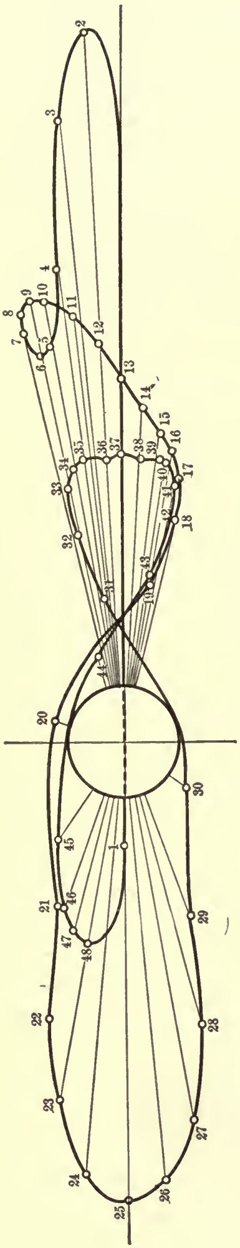


FIG. 325.—Wrist Pin Pressure.

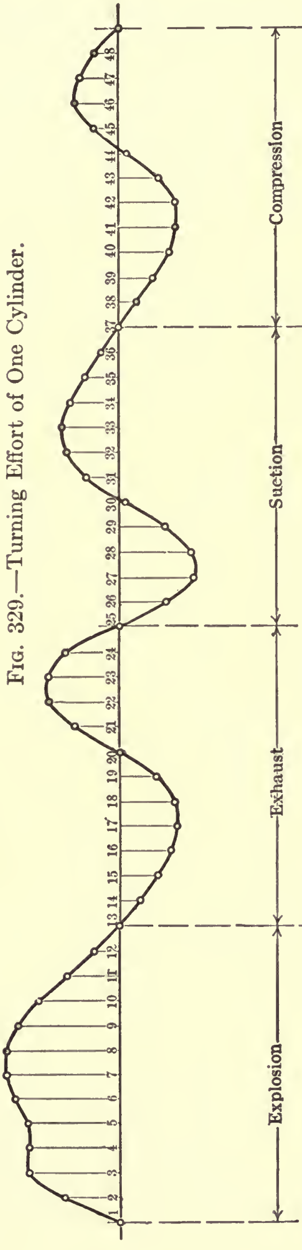


FIG. 329.—Turning Effort of One Cylinder.

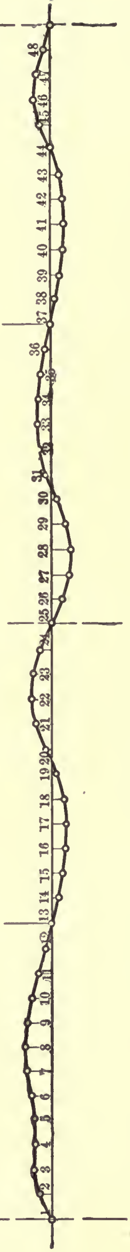


FIG. 326.—Pressure on Cylinder Wall.

of Art. 93 tabulated in Tables VI to IX, and plotted to a scale 1 inch = 250 pounds in Fig. 325. In this figure the wrist pin is drawn full size, and the pressures are plotted as vectors measured outward from the circumference. Fig. 325 shows plainly the portions of the pin where wear is to be expected.

The side thrust on the cylinder wall is determined as in Art. 93, entered in the last column of Tables V to IX, and plotted in Fig. 326.

Crank-pin Pressure. Bearing Pressure. Turning Effort.—The crank-pin pressure is determined as in Art. 93, entered in Tables VI to IX, and plotted in Fig. 327. In this figure the crank

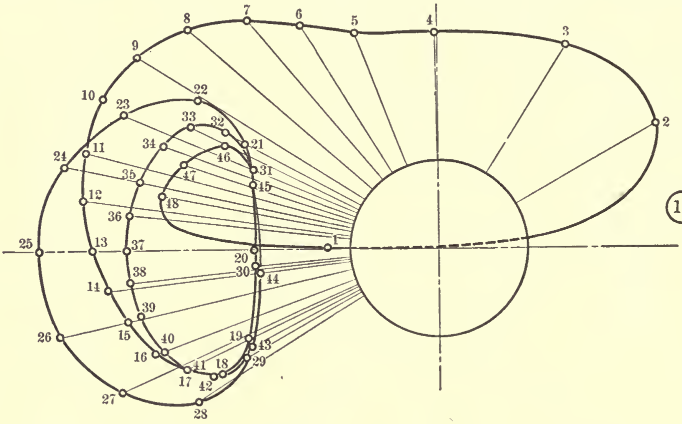


FIG. 327.—Crank Pin Pressures.

pin is drawn full size and the pin pressures are drawn as vectors measured outward from the circumference. As the crank-pin is rotating the vectors are drawn, not in their true direction, but in their direction relative to a line turning with the pin. In this way it is clearly indicated that the pressure is always on the same side of the pin, thus demonstrating the cause of uneven wear of the pin.

The crank-pin pressure is resolved into radial and tangential components. The radial component gives the pressure on the main bearing. The values of the bearing pressure are entered in Tables VI to IX, and plotted in Fig. 328. In this figure the journal is drawn full size, and vectors are drawn outward from the

circumference representing the pressure in magnitude and direction.

The tangential component of the crank-pin pressure is the *turning effort.* This is entered in Tables VI to IX, and is plotted in Fig. 329 on the developed crank circle as a base. A turning effort which assists the motion is plotted above the base line, while one which tends to hinder the motion is plotted below the base line.

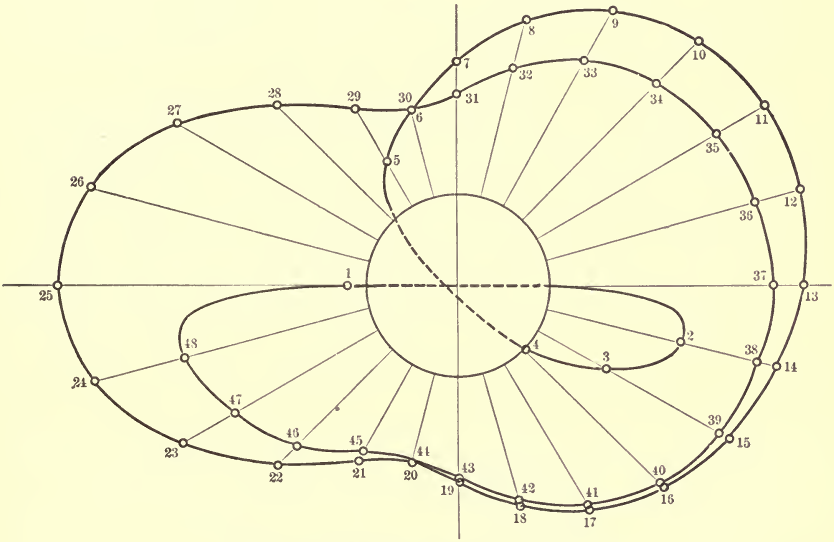


Fig. 328.—Main Bearing Pressure.

Combined Turning Effort.—All the forces so far considered have been those developed in a single cylinder, rod, and crank. The total turning effort of the engine is the resultant of the efforts of the separate cylinders. In Fig. 330 is shown the order in which events occur in the different cylinders. With the aid of this diagram it is possible to lay out a common base line for the turning efforts of all the cylinders and to plot the combined effort as shown in Fig. 331. In Fig. 331, the ordinate is the turning effort and the abscissa the movement of the crank. The area under the curve is therefore the work performed. When the curve rises above the mean ordinate excess work is performed

and the engine speeds up. When the curve falls below the mean ordinate the engine slows down.

Shaking Forces.—The centrifugal force of the crank arm and pin, the inertia force of the rod, and the inertia force of the piston all tend to cause vibrations of the engine. In Fig. 332 the shaking forces for a single cylinder are shown. From a pole O the vector OA is laid off representing the centrifugal force of the crank arm and pin. From A is drawn AB representing the inertia force of the rod, and from B is drawn BC equal to the inertia force of the piston. The resultant OC is the total shaking force. The vectors are laid off for every 15° of crank travel and a smooth

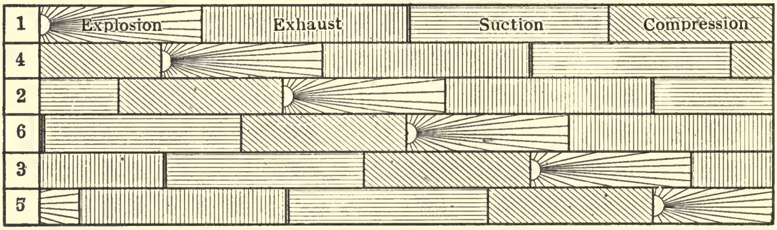


Fig. 330.—Diagram Showing Sequence of Functions.

curve R is drawn through the ends of the vectors. The radius of the large circle represents the mean value of the shaking force. If a counterweight is placed on the crank of size sufficient to balance the centrifugal force of the crank arm and pin, the remaining shaking forces are only the inertia forces of the rod and piston. Curve T, Fig. 332, shows the reduction of shaking force due to counterbalancing the crank. If a still larger counterweight is used such that the centrifugal force of this weight equals the mean shaking force, the result is shown in curve S. It is evident, therefore that by suitable counterweights the shaking forces can be greatly diminished.

It should be noted that curves R, S and T give the shaking forces for a single cylinder. The shaking forces for all six cylinders form a balanced system, see Art. 115.

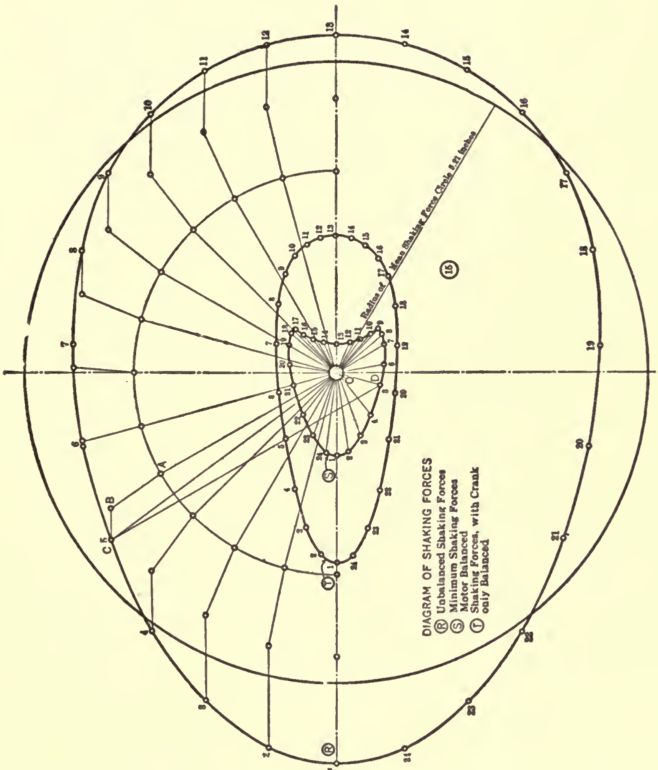


DIAGRAM OF SHAKING FORCES
R Unbalanced Shaking Forces
S Minimum Shaking Forces
S Motor Balanced
T Shaking Forces, with Crank only Balanced

FIG. 332.

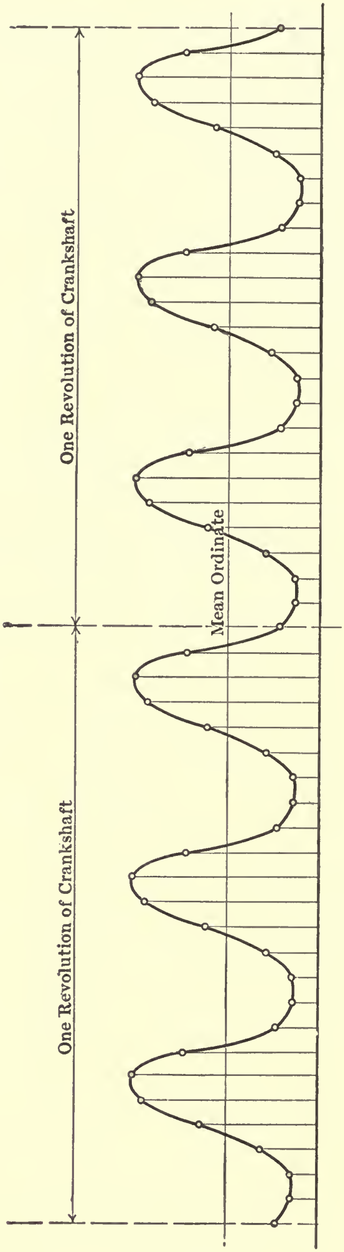


Fig. 331.—Combined Turning Effort.

TABLE I

COMPRESSION CURVE

No.	V	Log V_1	Log $\frac{V}{V_1}$	1.28 log $\frac{V}{V_1}$	Log P_1	P_1	PA	$\frac{PA}{500}$
1	1.294	.11193	......	......	1.16732	14.7	112.7	.225
2	1.194	.07700	.03493	.04471	1.21203	16.29	125.0	.25
3	1.094	.03902	.03798	.04861	1.26064	18.22	139.8	.279
4	.994	$\bar{1}$.99739	.04163	.05329	1.31393	20.60	158.0	.316
5	.894	$\bar{1}$.95134	.04605	.05894	1.37287	23.60	181.0	.36
6	.794	$\bar{1}$.8 982	.05158	.06595	1.43882	27.47	210.7	.42
7	.694	$\bar{1}$.84136	.05846	.07483	1.51365	32.63	250.3	.50
8	.594	$\bar{1}$.77379	.06757	.08649	1.60014	39.82	305.5	.61
9	.494	$\bar{1}$.69373	.08006	.10248	1.70262	50.42	386.8	.77
10	.394	$\bar{1}$.59550	.09823	.12573	1.82835	67.35	516.7	1.03
11	.294	$\bar{1}$.46835	.12715	.16275	1.99110	97.97	751.5	1.50

Drawn H. A. Gulley	TABLE I	Date May 28, 1920
Traced H. A. Gulley	FALLS MOTOR MODEL "S"	Scale
Checked Edmonds	Coll. of Eng.—Univ. of Ill.	Order No.
Approved A. C. H.	Mech. Eng. Dept.	Sheet No.

TABLE II

CALCULATION OF VALUES FOR PLOTTING EXPANSION CURVE

No.	V_1	Log V_1	Log $\frac{V}{V_1}$	1.23 log $\frac{V}{V_1}$	Log P_1	P_1	PA	$\frac{PA}{500}$
1	.294	$\bar{1}$.46835	......	......	2.59318	391.90	3006.0	6.01
2	.394	$\bar{1}$.59550	$\bar{1}$.87285	$\bar{1}$.84361	2.43679	273.40	209.50	4.19
3	.494	$\bar{1}$.69373	$\bar{1}$.90177	$\bar{1}$.87918	2.31597	207.00	1588.0	3.17
4	.594	$\bar{1}$.77379	$\bar{1}$.91991	$\bar{1}$.90153	2.21750	165.00	1266.0	2.53
5	.694	$\bar{1}$.84136	$\bar{1}$.93243	$\bar{1}$.91689	2.13439	136.30	1045.0	2.09
6	.794	$\bar{1}$.89982	$\bar{1}$.94154	$\bar{1}$.92809	2.06248	115.50	885.7	1.77
7	.894	$\bar{1}$.95134	$\bar{1}$.94848	1.93663	1.99911	99.80	765.5	1.53
8	.994	$\bar{1}$.99739	$\bar{1}$.95395	$\bar{1}$.94336	1.94247	87.59	671.9	1.34
9	1.094	.03902	$\bar{1}$.95837	$\bar{1}$.94880	1.89127	77.85	597.1	1.19
10	1.194	.07700	$\bar{1}$.96202	$\bar{1}$.95328	1.84455	69.92	536.1	1.07
11	1.294	.11193	$\bar{1}$.96507	$\bar{1}$.95704	1.80159	63.33	485.7	97

Drawn Edmonds	TABLE II	Date May 24, 1920
Traced Edmonds	BALANCING PROBLEM	Scale
Checked Edmonds	College of Engineering U. of Illinois	Order No.
Approved A. C. H.	Department of Mechanical Engineering	Sheet No.

TABLE III

Calculation of Acceleration of the Piston

Piston Position	Angle θ	$\cos\theta$	2θ	$\cos 2\theta$	$\frac{r}{L}\cos 2\theta$	$\cos\theta+\frac{r}{L}\cos 2\theta$	$a=r\omega^2\left(\cos\theta+\frac{r}{L}\cos 2\theta\right)$
1	0 360	1.0	0 720	1.0	.2615	1.2615	15,315
2 24	15 345	.9659	30 690	.8660	.2264	1.1923	14,475
3 23	30 330	.8660	60 660	.50	.13075	.99675	12,100
4 22	45 315	.7071	90 630	0.0	0.0	.7071	8,583
5 21	60 300	.50	120 600	−.50	−.13075	.36925	4,483
6 20	75 285	.2588	150 570	−.8660	−.2264	.0324	393
7 19	90 270	0.0	180 540	−1.0	−.2615	−.2615	−3,174
8 18	105 255	−.2588	210 510	−.8660	−.2264	−.4852	−5,890
9 17	120 240	−.50	240 480	−.50	−.13075	−.63075	−7,657
10 16	135 225	−.7071	270 450	0.0	0.0	−.7071	−8,583
11 15	150 210	−.8660	300 420	.50	.13075	−.73525	−8,925
12 14	165 195	−.9659	330 390	.8660	.2264	−.7395	−8,977
13	180	−1.0	360	1.0	.2615	−.7385	−8,965

Drawn Staley	TABLE III	Date May 24, 1920
Traced Staley	Balancing Problem	Scale
Checked Gulley	Coll. of Eng. Univ. of Ill.	Order No.
Approved A. C. H.	Mech. Eng. Dept.	Sheet No.

TABLE IV

INERTIA FORCES OF THE PISTON

Piston Position	Vector OB in Inches	Acc. of Piston = $OB \times 5711.5$	Force of Inertia = Ma	$\frac{F}{500}$
1	2.68	15,307	921.5	1.843
2 24	2.535	14,478	871.5	1.743
3 23	2.12	12,108	728.9	1.457
4 22	1.52	8,581	517.5	1.035
5 21	.785	4,484	269.9	.539
6 20	.07	399	24.02	.048
7 19	−.57	−3,155	189.9	−.379
8 18	−1.035	−5,911	355.8	−.711
9 17	−1.34	−7,653	460.7	−.921
10 16	−1.50	−8,467	509.6	−1.019
11 15	−1.55	−8,742	526.3	−1.052
12 14	−1.57	−8,867	533.8	−1.068
13	−1.575	−8,894	535.4	−1.071

Drawn Staley	TABLE V	Date May 24, 1920
Traced Staley	BALANCING PROBLEM	Scale
Checked Edmonds	Coll. of Eng. Univ. of Ill.	Order No.
Approved A. C. H.	Mech. Eng. Dept.	Sheet No.

TABLE V

Calculation of Inertia Forces of Connecting Rod "Q" Forces

Piston Positions		$r=OC$ Inches	OG Inches	Acceleration $OG \times 5711.5$	$F=Ma$ $M=.0544$	$\frac{F}{500}$	$\frac{3F}{500}$
1		2.125	2.25	12,840	698.5	1.397	4.191
2	24	2.125	2.21	12,622	686.6	1.373	4.119
3	23	2.125	2.08	11,880	646.3	1.293	3.879
4	22	2.125	1.90	10,852	590.3	1.181	3.543
5	21	2.125	1.74	9,938	540.6	1.081	3.243
6	20	2.125	1.66	9,481	515.7	1.031	3.093
7	19	2.125	1.65	9,424	502.6	1.005	3.015
8	18	2.125	1.72	9,823	534.3	1.069	3.207
9	17	2.125	1.82	10,394	565.4	1.131	3.393
10	16	2.125	1.90	10,851	590.3	1.181	3.543
11	15	2.125	1.96	11,194	609.9	1.220	3.660
12	14	2.125	1.99	11,365	618.3	1.237	3.711
13		2.125	2.00	11,423	621.4	1.243	3.729

Drawn Houser	TABLE IV		Date May 25, 1920
Traced Houser	Balancing Problem		Scale
Checked Edmonds	Coll. of Eng.	Univ. of Ill.	Order No.
Approved A. C. H.	Mech. Eng. Dept.		Sheet No.

TABLE VI

FORCES ACTING DURING EXPLOSION STROKE

All values given are vector lengths in inches. Scale 1 inch = 500 lb.

No.	Position of Crank in Degrees	Q_1 Force at Wrist Pin *AF*	Q_2 Force at Crank Pin *FK*	Total Force at Wrist Pin *AL*	Total Force at Crank Pin *KL*	Turning Effort *KP*	Main Bearing Pressure *LP*	Connecting Rod Thrust *FL*	Horizontal Thrust of Piston *BL*
1	0	.16	1.23	.19	1.23	0	1.23	1.23	0
2	15	.15	1.22	2.99	1.69	.84	1.47	2.83	.16
3	30	.12	1.17	2.59	1.61	1.37	.83	2.47	.27
4	45	.12	1.06	1.91	1.37	1.37	.03	1.82	.26
5	60	.09	.99	1.58	1.49	1.39	.54	1.54	.28
6	75	.07	.96	1.56	1.81	1.55	.93	1.56	.32
7	90	.06	.94	1.66	2.19	1.69	1.48	1.68	.38
8	105	.06	1.01	1.76	2.54	1.67	1.90	1.79	.41
9	120	.05	1.08	1.81	2.78	1.51	2.33	1.84	.38
10	135	.04	1.14	1.79	2.89	1.19	2.63	1.82	.21
11	150	.02	1.20	1.69	2.87	.79	2.77	1.73	.21
12	165	.00	1.24	1.56	2.79	.38	2.77	1.56	.09

Drawn Edmonds	TABLE VI		Date May 24, 1920
Traced Edmonds	BALANCING PROBLEM		Scale 500 lb. = 1 in.
Checked Staley	College of Engineering	U. of Illinois	Order No.
Approved A. C. H.	Department of Mechanical Engineering		Sheet No.

TABLE VII

FORCES ACTING DURING EXHAUST STROKE

All values given are vector lengths in inches. Scale 1 inch – 500 lb.

No.	Guide Bar Pressure *BL*	Q_1 at Wrist Pin	Q_2 at Crank Pin	Wrist Pin Pressure *AL*	Connecting Rod Thrust *FL*	Crank Pin Pressure *KL*	Turning Effort *KP*	Pressure on Main Bearing *LP*
13	.00	.00	1.25	1.41	1.41	2.67	.00	2.67
14	.08	.02	1.22	1.27	1.29	2.50	.33	2.49
15	.15	.02	1.20	1.16	1.18	2.34	.55	2.28
16	.19	.03	1.15	1.10	1.11	2.21	.79	2.06
17	.21	.03	1.10	.97	1.00	1.97	.86	1.76
18	.16	.04	1.01	.78	.81	1.63	.82	1.40
19	.08	.04	.96	.46	.48	1.21	.53	1.10
20	.06	.06	.97	.07	.03	.97	.00	.97
21	.21	.09	1.00	.55	.58	1.34	.64	1.18
22	.27	.10	1.08	1.04	1.11	2.00	1.06	1.70
23	.26	.15	1.14	1.43	1.55	2.60	1.07	2.36
24	.16	.15	1.22	1.72	1.84	3.05	.69	2.98

Drawn Staley	TABLE VII	Date May 24, 1920
Traced Staley	BALANCING PROBLEM	Scale
Checked Edmonds	Coll. of Eng. Univ. of Ill.	Order No.
Approved A. C. H.	Mech. Eng. Dept.	Sheet No.

TABLE VIII

FORCES ACTING DURING SUCTION STROKE

All values given are vector lengths in inches. Scale 1 inch = 500 lb.

No.	Position	Q_1 at Wrist Pin	Q_2 at Crank Pin	Wrist Pin Pressure *AL*	Connecting Rod Thrust *FL*	Crank Pin Pressure *KL*	Turning Effort *KP*	Pressure on Main Bearing *LP*	Guide Bar Pressure *BL*
25	Suction	0	1.40	1.83	1.83	3.24	0	3.24	0
26	"	.13	1.24	1.77	1.90	3.10	.71	3.02	.16
27	"	.14	1.15	1.50	1.62	2.67	1.11	2.42	.27
28	"	.12	1.06	1.09	1.16	2.03	1.09	1.71	.29
29	"	.08	1.00	.60	.63	1.37	.70	1.18	.22
30	"	.07	.96	.10	.07	.97	.09	.96	.08
31	"	.05	.96	.40	.42	1.16	.47	1.06	.06
32	"	.04	1.03	.71	.74	1.59	.77	1.39	.15
33	"	.04	1.09	.92	.95	1.93	.85	1.74	.19
34	"	.04	1.14	1.01	1.04	2.12	.73	1.99	.18
35	"	0	1.22	1.04	1.04	2.23	.49	2.18	.14
36	"	.01	1.23	1.05	1.07	2.28	.25	2.27	.07

Drawn Houser	TABLE VIII	Date May 26, 1920
Traced Houser	BALANCING PROBLEM	Scale
Checked Edmonds	Coll. of Eng. Univ. of Ill.	Order No.
Approved A. C. H.	Mech. Eng. Dept.	Sheet No.

TABLE IX

Forces Acting during Compression Stroke

All values given are vector lengths in inches. Scale 1 inch = 500 lb.

No.	Position Degrees	Q_1 at Wrist Pin *AF*	Q_2 at Crank Pin *FK*	Wrist Pin Pressure *AL*	Crank Pin Pressure *KL*	Turning Effort *KP*	Main Bearing Pressure *LP*	Connecting Rod Thrust *FL*	Horizontal Cylinder Thrust *BL*
37	180	.00	1.25	1.07	2.30	.00	2.30	1.07	.00
38	195	.00	1.24	1.04	2.28	.24	2.26	1.04	.07
39	210	.00	1.22	1.04	2.24	.49	2.19	1.04	.16
40	225	.025	1.155	1.035	2.16	.74	2.00	1.06	.16
41	240	.03	1.10	.95	1.94	.85	1.75	.97	.19
42	255	.04	1.01	.79	1.65	.84	1.44	.80	.19
43	270	.07	.93	.51	1.27	.59	1.11	.53	.10
44	285	.09	.94	.15	.95	.11	.96	.13	.04
45	300	.10	.99	.27	1.11	.37	1.06	.29	.16
46	315	.13	1.05	.56	1.56	.66	1.41	.64	.22
47	330	.13	1.16	.64	1.83	.58	1.73	.73	.16
48	345	.14	1.23	.66	1.99	.35	1.97	.82	.11

Drawn Gulley	TABLE IX	Date May 28, 1920
Traced H. A. G.	Balancing Problem	Order No.
Checked Edmonds	Coll. of Eng. Univ. of Ill.	Scale 1 inch = 500 lb.
Approved A. C. H.	Mech. Eng. Dept.	Sheet No.

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